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2009 J. Phys. A: Math. Theor. 42 085204

(http://iopscience.iop.org/1751-8121/42/8/085204)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 085204 (21pp) doi:10.1088/1751-8113/42/8/085204

Invariants of a family of third-order ordinary differential equations

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Received 13 August 2008, in final form 25 November 2008
Published 2 February 2009
Online at stacks.iop.org/JPhysA/42/085204

Abstract
We consider a family of third-order ordinary differential equations (ODEs) with the cubic nonlinearity in second-order derivative, which is closed with respect to contact transformations. Using Lie's infinitesimal method we construct the basis of differential invariants for this family of equations and the operators of invariant differentiation. Invariants provide a simple way of finding the equations, which may be equivalent to a given nonlinear ODE and the transformation connecting two equivalent equations. Some examples are given to illustrate the results obtained in the paper.

PACS numbers: 02.30.Hq, 02.30.Ik

1. Introduction

It was noted by Lie [1] that the family of third-order ordinary differential equations (ODEs)

$$y''' = S(x, y, y')y''^3 + 3R(x, y, y')y''^2 + 3Q(x, y, y')y'' + P(x, y, y') \tag{1}$$

is closed with respect to an arbitrary invertible contact change of variables

$$\bar{x} = \Phi(x, y, y'), \qquad \bar{y} = \Psi(x, y, y'), \qquad \bar{y}' = \Omega(x, y, y'), \tag{2}$$

$$\Omega = D\Psi/D\Phi, \qquad (\Psi_x + y'\Psi_y)\Phi_{y'} - (\Phi_x + y'\Phi_y)\Psi_{y'} = 0. \tag{3}$$

Here $D = \partial_x + y'\partial_y + y''\partial_{y'} + \cdots$ is the operator of total derivative and the subscripts stand for partial differentiation.

Transformation (2) is an equivalence transformation for equation (1). It relates equation (1) to an equation of the same form

$$\bar{y}''' = \bar{S}(\bar{x}, \bar{y}, \bar{y}')\bar{y}''^3 + 3\bar{R}(\bar{x}, \bar{y}, \bar{y}')\bar{y}''^2 + 3\bar{Q}(\bar{x}, \bar{y}, \bar{y}')\bar{y}'' + \bar{P}(\bar{x}, \bar{y}, \bar{y}'), \tag{4}$$

where the functional dependence of the coefficients $\bar{P}$, $\bar{Q}$, $\bar{R}$, $\bar{S}$ on $\bar{x}$, $\bar{y}$, $\bar{y}'$ may be different from that of P, Q, R, S on x, y, y' in the ODE (1). Similarity reductions of partial differential equations (PDEs) often lead to ODEs of the form (1). However, it is not always evident how to integrate these ODEs. One way is to find another equation (4) with a known solution, which is related to the given ODE (1) by an invertible transformation (2). This equivalence problem may be solved by use of the invariants of the group of contact equivalence transformations of the family of equations (1). These invariants are usually referred to simply as invariants of the corresponding family of equations. Equations (1) and (4) are equivalent with respect to a contact transformation (2) iff all their invariants are equal.

There are two systematic approaches to the equivalence problem for differential equations: Lie's infinitesimal method and Cartan's equivalence method. But most of the authors applying Cartan's method [2] do not provide explicit formulas for invariants of the families of equations under study, except perhaps the paper [3], where the equivalence problem is solved explicitly.

Lie's infinitesimal method is more widely used [4]. Based on ideas outlined by Lie [5], this approach has been developed in Ovsyannikov's book [6] for infinite Lie groups and has been applied later in [7] to equivalence transformation groups. The group of transformations can possess infinitely many differential invariants [6]. As follows from [6, 8] there exists the finite basis of differential invariants such that all other higher order invariants of the group are obtained from the basis ones by algebraic operations and invariant differentiations. The order of an invariant I is defined by the highest order of derivatives of functions P, Q, R, S with respect to x, y, y' involved in I. Operators $\mathcal{D}$ of invariant differentiation satisfy the condition: if I is an invariant of equation (1), then $\mathcal{D}I$ is its invariant too. The number of independent operators of invariant differentiation coincides with the number of variables on which depend the arbitrary elements (functions P, Q, R, S) in equation (1).

The linearization problem is a particular case of the equivalence problem when one of equations (1) and (4) is linear. It has been studied in detail in [9, 10]. Linearization criteria for the third-order ODE (1) have been obtained in [9] by Cartan's method and then in [10] by direct approach. The problem of equivalence to Chazy equations with respect to point transformations (when $\Phi_{y'} \equiv 0$, $\Psi_{y'} \equiv 0$ in (2)) is solved in [11] for a particular case of equation (1) corresponding to $S(x, y, y') \equiv 0$.

The outline of the present paper is as follows. In section 2 we apply Lie's method to construct the basis of third-order invariants of equation (1) and find three independent operators of invariant differentiation. In section 3 several examples show how the invariants can be applied in solving the equivalence problem. Concluding remarks are made in section 4.

In the calculations presented here, we have used computer algebra tools for checking the results obtained and some tedious computations. Finally, expressions (7)–(12) have been written in the Maple package, and the invariants in all examples of section 3 have been obtained by substituting the concrete functions $P(x, y, y')$, $Q(x, y, y')$, $R(x, y, y')$, $S(x, y, y')$ into these formulas.

2. Invariants of equation (1)

The nth-order differential invariant I of equation (1) is found from the invariance criterion $\widetilde{X} I = 0$ [6], where $\widetilde{X}$ is the generator X of the group of contact equivalence transformations of equation (1) extended to the derivatives of P, Q, R, S up to the nth order. In our study of invariants of equation (1), we use two results established in [10], which may be stated as follows.

Lemma 1. *Generator X of the group of contact equivalence transformations of equation (1) has the form*

$$X = -w_{y'}\partial_x + (w - y'w_{y'})\partial_y + (w_x + y'w_y)\partial_{y'} + \pi\partial_P + \kappa\partial_Q + \rho\partial_R + \sigma\partial_S, \tag{5}$$

where $w = w(x, y, y')$ is an arbitrary function known as the characteristic function of the group of contact transformations (2), π, κ, ρ, σ depend on x, y, y', P, Q, R, S as

$$\begin{aligned}
&\pi = \mathrm{d}^3 w - 3Q\,\mathrm{d}^2 w + P(3\,\mathrm{d}(w_{y'}) + w_y), \qquad \sigma = w_{y'y'y'} - S(3\,\mathrm{d}(w_{y'}) + 2w_y) + 3Rw_{y'y'},\\
&\kappa = \mathrm{d}^2(w_{y'}) + \mathrm{d}(w_y) - 2R\,\mathrm{d}^2 w + Q\,\mathrm{d}(w_{y'}) + Pw_{y'y'},\\
&\rho = \mathrm{d}(w_{y'y'}) + w_{yy'} - S\,\mathrm{d}^2 w - R(\mathrm{d}w)_{y'} + 2Qw_{y'y'}, \qquad \mathrm{d} = \partial_x + y'\partial_y.
\end{aligned} \tag{6}$$

Lemma 2. *Equation (1) does not have nontrivial algebraic invariants and differential invariants of the first and the second order.*

Therefore, we should search the third-order invariants

$$I(x, y, y', P, Q, R, S, P_x, P_y, P_{y'}, \ldots, S_{y'}, P_{xx}, \ldots, S_{y'y'}, P_{xxx}, \ldots, S_{y'y'y'})$$

of equation (1). For the most compact representation of results, we use so-called pseudovector fields [12], which depend on the derivatives of P, Q, R, S of the first order

$$\begin{aligned}
&\alpha = S_x + y'S_y - R_{y'} + 2QS - 2R^2, \qquad \beta = R_x + y'R_y - Q_{y'} + PS - QR,\\
&\gamma = Q_x + y'Q_y - P_{y'} + 2PR - 2Q^2,
\end{aligned} \tag{7}$$

second order

$$\begin{aligned}
&A_1 = \alpha_{y'} + 2R\alpha - 2S\beta + 2S_y, && A_2 = \alpha_x + y'\alpha_y + 2Q\alpha - 2R\beta + 2R_y,\\
&B_1 = \beta_{y'} + Q\alpha - S\gamma + 2R_y, && B_2 = \beta_x + y'\beta_y + P\alpha - R\gamma + 2Q_y,\\
&\Gamma_1 = \gamma_{y'} + 2Q\beta - 2R\gamma + 2Q_y, && \Gamma_2 = \gamma_x + y'\gamma_y + 2P\beta - 2Q\gamma + 2P_y,\\
&A_0 = A_2 + 2B_1, && B_0 = 2B_2 + \Gamma_1,
\end{aligned} \tag{8}$$

third order

$$\begin{aligned}
&M_1 = A_{1y'} + 3RA_1 - SA_0, && M_2 = A_{1x} + y'A_{1y} + 3QA_1 - RA_0,\\
&M_3 = A_{0y'} + 3QA_1 + RA_0 - 2SB_0, && M_4 = A_{0x} + y'A_{0y} + 3PA_1 + QA_0 - 2RB_0,\\
&M_5 = B_{0y'} + 2QA_0 - RB_0 - 3S\Gamma_2, && M_6 = B_{0x} + y'B_{0y} + 2PA_0 - QB_0 - 3R\Gamma_2,\\
&M_7 = \Gamma_{2y'} + QB_0 - 3R\Gamma_2, && M_8 = \Gamma_{2x} + y'\Gamma_{2y} + PB_0 - 3Q\Gamma_2,\\
&L_1 = A_{1y} + \tfrac{1}{2}(\alpha A_0 - 3\beta A_1), && L_2 = A_{0y} + \alpha B_0 - \tfrac{1}{2}(\beta A_0 + 3\gamma A_1),\\
&L_3 = B_{0y} + \tfrac{1}{2}(3\alpha\Gamma_2 + \beta B_0) - \gamma A_0, && L_4 = \Gamma_{2y} + \tfrac{1}{2}(3\beta\Gamma_2 - \gamma B_0)
\end{aligned} \tag{9}$$

and their combinations: the relative invariants

$$\begin{aligned}
&a = 3A_1B_0 - A_0^2, \qquad b = 9A_1\Gamma_2 - A_0B_0, \qquad c = 3A_0\Gamma_2 - B_0^2,\\
&j_0 = 3cL_1 - bL_2 + aL_3 + \tfrac{3}{2}A_1(3M_2M_8 + M_4(M_7 - M_6) + M_5(M_6 - 2M_7))\\
&\quad + \tfrac{1}{2}B_0(3M_1M_6 + 6M_2(M_5 - M_4) + M_3(M_4 - 2M_5)) + \tfrac{3}{2}A_0((M_3 - 2M_2)M_7 - M_1M_8)\\
&\quad + \tfrac{1}{2}A_0\big(2M_4^2 - 2M_4M_5 + M_5^2 - M_3M_6\big) + \tfrac{3}{2}\Gamma_2\big(6M_2^2 - 3M_2M_3 + M_3^2 - M_1(M_4 + M_5)\big),
\end{aligned}$$

$$
\begin{aligned}
j_1 &= cL_2 - bL_3 + 3aL_4 + \tfrac{3}{2}\Gamma_2((2M_2 - M_3)M_4 + (M_3 - M_2)M_5 - 3M_1M_7) \\
&\quad + \tfrac{1}{2}A_0(M_6(2M_4 - M_5) + 6M_7(M_5 - M_4) - 3M_3M_8) + \tfrac{3}{2}B_0(M_1M_8 + M_2(2M_7 - M_6)) \\
&\quad + \tfrac{1}{2}B_0\big(M_3M_6 - M_4^2 + 2M_4M_5 - 2M_5^2\big) + \tfrac{3}{2}A_1\big((M_4 + M_5)M_8 - M_6^2 + 3M_6M_7 - 6M_7^2\big), \\
j_3 &= A_0M_1 - 3A_1M_2, \qquad j_4 = 2B_0M_1 + A_0(M_3 - 2M_2) - 3A_1M_4, \\
j_5 &= 3\Gamma_2M_1 + B_0(2M_3 - M_2) + A_0(M_5 - 2M_4) - 3A_1M_6, \\
j_6 &= 3\Gamma_2M_3 + B_0(2M_5 - M_4) + A_0(M_7 - 2M_6) - 3A_1M_8, \\
j_7 &= 3\Gamma_2M_5 + B_0(2M_7 - M_6) - 2A_0M_8, \qquad j_8 = 3\Gamma_2M_7 - B_0M_8, \\
i_1 &= 10B_0j_3 - 4A_0j_4 + 3A_1j_5, \qquad i_2 = 30\Gamma_2j_3 + 2B_0j_4 - 5A_0j_5 + 9A_1j_6, \\
i_3 &= 6\Gamma_2j_4 - B_0j_5 - A_0j_6 + 6A_1j_7, \qquad i_4 = 9\Gamma_2j_5 - 5B_0j_6 + 2A_0j_7 + 30A_1j_8, \\
i_5 &= 3\Gamma_2j_6 - 4B_0j_7 + 10A_0j_8, \\
k_1 &= 4\Gamma_2i_1 - B_0i_2 + 2A_0i_3 - A_1i_4, \qquad k_2 = \Gamma_2i_2 - 2B_0i_3 + A_0i_4 - 4A_1i_5, \\
k_3 &= 5(3\Gamma_2b - 2B_0c)M_1 + 3(B_0b - 6\Gamma_2a)(2M_2 + M_3) \\
&\quad + (A_0b - 6A_1c)(4M_4 + M_5) + (3A_1b - 2A_0a)(2M_6 - M_7), \\
k_4 &= (3\Gamma_2b - 2B_0c)(2M_3 - M_2) + (B_0b - 6\Gamma_2a)(M_4 + 4M_5) \\
&\quad + 3(A_0b - 6A_1c)(M_6 + 2M_7) + 5(3A_1b - 2A_0a)M_8
\end{aligned}
\tag{10}
$$

and

$$
\begin{aligned}
J_1 &= c(3M_2 - M_3) + b(M_5 - M_4) + a(M_6 - 3M_7), \qquad J_2 = 3\Gamma_2L_1 - B_0L_2 \\
&\quad + A_0L_3 - 3A_1L_4 - M_1M_8 - 2M_2M_7 + M_2M_6 + M_3M_7 + \big(M_4M_5 - M_4^2 - M_5^2\big)/3, \\
J_3 &= 2c^2M_1 - bc(M_2 + M_3) + (b^2 + 2ac)(M_4 + M_5)/3 - ab(M_6 + M_7) + 2a^2M_8, \\
J_4 &= 4i_1i_5 - i_2i_4 + 3i_3^2, \qquad J_5 = cj_0^2 - bj_0j_1 + aj_1^2, \\
J_6 &= c\big(20j_3j_7 - 8j_4j_6 + 3j_5^2\big) + b(6j_4j_7 - 50j_3j_8 - j_5j_6) + a\big(20j_4j_8 - 8j_5j_7 + 3j_6^2\big), \\
J_7 &= i_1i_4^2 - 8i_1i_3i_5 + i_2^2i_5 - i_2i_3i_4 + 2i_3^3, \qquad J_8 = \Gamma_2j_0^3 - B_0j_0^2j_1 + A_0j_0j_1^2 - A_1j_1^3, \\
J_9 &= 3\Gamma_2\big(10j_3j_6^2 - 20j_3j_5j_7 - 4j_4j_5j_6 + 8j_4^2j_7 + j_5^3\big) \\
&\quad + B_0\big(100j_3j_5j_8 - 20j_3j_6j_7 - 40j_4^2j_8 + 4j_4j_5j_7 + 2j_4j_6^2 - j_5^2j_6\big) \\
&\quad + A_0\big(20j_4j_5j_8 - 100j_3j_6j_8 + 40j_3j_7^2 - 4j_4j_6j_7 - 2j_5^2j_7 + j_5j_6^2\big) \\
&\quad + 3A_1\big(20j_4j_6j_8 - 10j_5^2j_8 + 4j_5j_6j_7 - 8j_4j_7^2 - j_6^3\big), \qquad J_0 = (b^2 - 4ac)/60.
\end{aligned}
\tag{11}
$$

Theorem 1. *Nine third-order invariants*

$$
\begin{aligned}
&I_1 = J_0^{-5/6}J_1, \quad I_2 = J_0^{-2/3}J_2, \quad I_3 = J_0^{-4/3}J_3, \quad I_4 = J_0^{-5/3}J_4, \\
&I_5 = J_0^{-7/3}J_5, \quad I_6 = J_0^{-5/3}J_6, \quad I_7 = J_0^{-5/2}J_7, \quad I_8 = J_0^{-3}J_8, \quad I_9 = J_0^{-2}J_9
\end{aligned}
\tag{12}
$$

form the basis of differential invariants for equation (1), the invariant differentiations are defined by the operators

$$
\begin{aligned}
&\mathcal{D}_1 = J_0^{-1}(j_0(D_x + y'D_y) - j_1D_{y'}), \qquad \mathcal{D}_2 = J_0^{-7/6}(k_1(D_x + y'D_y) - k_2D_{y'}), \\
&\mathcal{D}_3 = J_0^{-7/6}(k_3(D_x + y'D_y) + 7J_0D_y - k_4D_{y'}).
\end{aligned}
\tag{13}
$$

Any other differential invariant of equation (1) is a function of invariants (12) and their invariant derivatives.

To prove this statement, we calculate the third extension $\widetilde{X}$ of generator (5)

$$\widetilde{X} = X + \pi^x \partial_{P_x} + \pi^y \partial_{P_y} + \pi^{y'} \partial_{P_{y'}} + \cdots + \sigma^{y'y'y'} \partial_{S_{y'y'y'}}$$

by the standard prolongation formulas [6] with x, y, y' regarded as independent variables and P, Q, R, S as dependent ones:

$$C^x = D_x C - T_x \xi_x - T_y \eta_x - T_{y'} \zeta_x, \qquad C^y = D_y C - T_x \xi_y - T_y \eta_y - T_{y'} \zeta_y,$$
$$C^{y'} = D_{y'} C - T_x \xi_{y'} - T_y \eta_{y'} - T_{y'} \zeta_{y'}. \tag{14}$$

Here C is equal to $\pi, \kappa, \rho, \sigma, \pi^x, \ldots, \sigma^{y'y'}$, while T is equal to $P, Q, R, S, P_x, \ldots, S_{y'y'}$ successively, $\xi = -w_{y'}, \eta = w - y'w_{y'}, \zeta = w_x + y'w_y$ are the coordinates of X at x, y, y', the functions $\pi, \kappa, \rho, \sigma$ are given by (6), $D_x = \partial_x + P_x \partial_P + \cdots$, $D_y = \partial_y + P_y \partial_P + \cdots$, $D_{y'} = \partial_{y'} + P_{y'} \partial_P + \cdots$ are the operators of total differentiation with respect to x, y, y'.

As is seen from (5), (6) and (14), the operator $\widetilde{X}$ has to depend linearly on the derivatives of an arbitrary function w up to the sixth order (their number is equal to 84). It can be represented in the form

$$\widetilde{X} = wX_1 + w_x X_2 + w_y X_3 + w_{y'} X_4 + w_{xx} X_5 + \cdots + \partial^6 w/\partial y'^6 X_{84}, \tag{15}$$

where the operators X_i are given in appendix A. According to the theory of invariants of infinite transformation groups [6], the relation $\widetilde{X} I = 0$ should be split by w and its derivatives. This gives rise to a homogeneous system of linear first-order PDEs

$$X_1 I = 0, \ldots, X_{84} I = 0. \tag{16}$$

Functionally independent solutions of system (16) provide the third-order invariants of equation (1).

The solution of system (16) is found in several steps. First, we solve the subsystem

$$X_1 I = 0, \qquad X_4 I = 0, \qquad X_{57} I = 0, \ldots, X_{84} I = 0 \tag{17}$$

with the most simple operators X_1, X_4 and 28 coefficients of the sixth-order derivatives of w in (15). Note that six of these operators are represented as linear functions of the remaining 24 operators (see appendix A). Therefore, in the space $R_{83}(x, y, y', P, Q, R, S, P_x, \ldots, S_{y'y'y'})$ system (17) has 59 independent solutions (here $d = \partial_x + y'\partial_y$)

$$y', P, Q, R, S, \qquad P_x, \ldots, S_{y'},$$
$$P_{xx}, \ldots, S_{y'y'}, \qquad u_1 = \mathrm{d}S_{y'y'} - R_{y'y'y'},$$
$$u_2 = \mathrm{d}^2 S_{y'} - \mathrm{d}R_{y'y'}, \qquad u_3 = \mathrm{d}^2 S_{y'} + \mathrm{d}R_{y'y'} - 2Q_{y'y'y'},$$
$$u_4 = \mathrm{d}^3 S + \mathrm{d}^2 R_{y'} - 2\,\mathrm{d}Q_{y'y'}, \qquad u_5 = 2\,\mathrm{d}^2 R_{y'} - \mathrm{d}Q_{y'y'} - P_{y'y'y'},$$
$$u_6 = 2\,\mathrm{d}^3 R - \mathrm{d}^2 Q_{y'} - \mathrm{d}P_{y'y'}, \qquad u_7 = \mathrm{d}^2 Q_{y'} - \mathrm{d}P_{y'y'}, \qquad u_8 = \mathrm{d}^3 Q - \mathrm{d}^2 P_{y'},$$
$$u_9 = \mathrm{d}S_{yy'} - R_{yy'y'}, \qquad u_{10} = \mathrm{d}^2 S_y + \mathrm{d}R_{yy'} - 2Q_{yy'y'},$$
$$u_{11} = 2\,\mathrm{d}^2 R_y - \mathrm{d}Q_{yy'} - P_{yy'y'}, \qquad u_{12} = \mathrm{d}^2 Q_y - \mathrm{d}P_{yy'},$$
$$u_{13} = \mathrm{d}^2 S_x - 5\,\mathrm{d}R_{xy'} + 4Q_{xy'y'}, \qquad u_{14} = \mathrm{d}R_{xy'} - Q_{xy'y'}, \qquad u_{15} = \mathrm{d}Q_{xy'} - P_{xy'y'},$$
$$u_{16} = \mathrm{d}S_{xx} - R_{xxy'}, \qquad u_{17} = \mathrm{d}R_{xx} - Q_{xxy'}, \qquad u_{18} = \mathrm{d}Q_{xx} - P_{xxy'}.$$

In these variables next 21 operators (coefficients of the fifth-order derivatives of w in (15)) become

$$
\begin{aligned}
X_{36} &= \partial_{P_{xx}} - S(2\partial_{u_6} + \partial_{u_{17}}) - 2R(\partial_{u_8} + \partial_{u_{18}}),\\
\bar{X}_{37} &= Z_P^1 + 2\partial_{u_8} + 2S(\partial_{u_{11}} - y'\partial_{u_{17}}) + 2R\partial_{u_{12}} + (2 - 4y'R)\partial_{u_{18}},\\
\bar{X}_{38} &= Z_P^2 - 2\partial_{u_{12}} - y'^2 S\partial_{u_{17}} + (4y' - 2y'^2 R)\partial_{u_{18}}, \qquad \bar{X}_{39} = 2y'^2\partial_{u_{18}},\\
X_{42} &= \partial_{Q_{xx}} + \partial_{P_{xy'}} - S(4\partial_{u_4} + 2\partial_{u_5} - 2\partial_{u_{13}} + \partial_{u_{14}} + 2\partial_{u_{16}}) - R(2\partial_{u_7} + 2\partial_{u_{15}} - \partial_{u_{17}})\\
&\quad + 4Q(\partial_{u_8} + \partial_{u_{18}}),\\
\bar{X}_{43} &= Z_Q^1 + Z_P^3 + 7\partial_{u_6} + 3\partial_{u_7} + Q(8y'\partial_{u_{18}} - 4\partial_{u_{12}}) + 4S\partial_{u_{10}}\\
&\quad + y'S(2\partial_{u_{13}} - \partial_{u_{14}} - 4\partial_{u_{16}}) + (3 - 2y'R)\partial_{u_{15}} + (2 + 2y'R)\partial_{u_{17}},\\
\bar{X}_{44} &= Z_Q^2 - 7\partial_{u_{11}} + 3y'\partial_{u_{15}} - 2y'^2 S\partial_{u_{16}} + (4y' + y'^2 R)\partial_{u_{17}} + 4y'^2 Q\partial_{u_{18}},\\
\bar{X}_{45} &= 2y'^2\partial_{u_{17}}, \qquad X_{47} = \partial_{R_{xx}} + \partial_{Q_{xy'}} + \partial_{P_{y'y'}} - S(2\partial_{u_2} + 4\partial_{u_3})\\
&\quad + R(6\partial_{u_4} + \partial_{u_{14}} + 4\partial_{u_{16}}) + Q(6\partial_{u_6} + 4\partial_{u_7} + 4\partial_{u_{15}} + \partial_{u_{17}}) - 2P(\partial_{u_8} + \partial_{u_{18}}),\\
\bar{X}_{48} &= Z_R^1 + Z_Q^3 + 8\partial_{u_4} + 10\partial_{u_5} + 2S\partial_{u_9} - 6R\partial_{u_{10}} - 6Q\partial_{u_{11}} + 2P\partial_{u_{12}} - 10\partial_{u_{13}}\\
&\quad + (3 + y'R)\partial_{u_{14}} + (2 + 8y'R)\partial_{u_{16}} + y'Q(4\partial_{u_{15}} + 2\partial_{u_{17}}) - 4y'P\partial_{u_{18}},\\
\bar{X}_{49} &= Z_R^2 - 8\partial_{u_{10}} - 10y'\partial_{u_{13}} + 3y'\partial_{u_{14}} + 4y'(1 + y'R)\partial_{u_{16}} + y'^2 Q\partial_{u_{17}} - 2y'^2 P\partial_{u_{18}},\\
\bar{X}_{50} &= 2y'^2\partial_{u_{16}}, \qquad X_{51} = \partial_{S_{xx}} + \partial_{R_{xy'}} + \partial_{Q_{y'y'}} - 2S\partial_{u_1} + R(4\partial_{u_2} + 6\partial_{u_3})\\
&\quad + Q(6\partial_{u_5} - 6\partial_{u_{13}} + \partial_{u_{14}} - 2\partial_{u_{16}}) - P(4\partial_{u_6} + 2\partial_{u_7} + 2\partial_{u_{15}} + \partial_{u_{17}}),\\
\bar{X}_{52} &= Z_S^1 + Z_R^3 + 3\partial_{u_2} + 11\partial_{u_3} - 4R\partial_{u_9} + 4P\partial_{u_{11}} + y'Q(\partial_{u_{14}} - 6\partial_{u_{13}} - 4\partial_{u_{16}})\\
&\quad - 2y'P(\partial_{u_{15}} + \partial_{u_{17}}), \qquad \bar{X}_{53} = Z_S^2 - 3\partial_{u_9} - 2y'^2 Q\partial_{u_{16}} - y'^2 P\partial_{u_{17}},\\
X_{54} &= \partial_{S_{xy'}} + \partial_{R_{y'y'}} + 4R\partial_{u_1} - 2Q\partial_{u_2} - P(2\partial_{u_4} + 4\partial_{u_5} - 4\partial_{u_{13}} + \partial_{u_{14}}),\\
\bar{X}_{55} &= Z_S^3 + 4\partial_{u_1} + 2Q\partial_{u_9} + P(2\partial_{u_{10}} + 4y'\partial_{u_{13}} - y'\partial_{u_{14}}),\\
X_{56} &= \partial_{S_{y'y'}} - 2Q\partial_{u_1} - 2P\partial_{u_3}.
\end{aligned}
$$

Here we use the notation from appendix A and omit three operators $\bar{X}_{40}, \bar{X}_{41}, \bar{X}_{46}$ since they are linear combinations of other operators. As 41 independent invariants of the operators $X_{36}, \ldots, X_{56}$ we can take

$$
\begin{aligned}
&y', P, Q, R, S, \qquad P_x, \ldots, S_{y'}, \qquad A_1, A_2, B_1, B_2, \Gamma_1, \Gamma_2,\\
&v_1 = S_{xy} + y'S_{yy} - R_{yy'}, \qquad v_2 = R_{xy} + y'R_{yy} - Q_{yy'},\\
&v_3 = Q_{xy} + y'Q_{yy} - P_{yy'}, \qquad M_1, \ldots, M_8, L_1, L_2, L_3, L_4,\\
&\bar{u}_{13} = u_{13} - 10R_{xy} - 2SP_{xy'} - 4PS_{xy'} + 6QR_{xy'},\\
&\bar{u}_{14} = u_{14} + 3R_{xy} + SP_{xy'} + PS_{xy'} - RQ_{xy'} - QR_{xy'},\\
&\bar{u}_{15} = u_{15} + 3Q_{xy} + 2(RP_{xy'} + PR_{xy'} - 2QQ_{xy'}),
\end{aligned}
\tag{18}
$$

where $A_1, A_2, B_1, B_2, \Gamma_1, \Gamma_2, M_i, L_j$ are defined by (8) and (9).

In variables (18) next 15 operators of system (16) take the form

$$
\begin{aligned}
X_{21} &= \partial_{P_x} + S_{y'}(4\partial_{\bar{u}_{13}} - \partial_{\bar{u}_{14}}) - 2R_{y'}\partial_{\bar{u}_{15}},\\
\bar{X}_{22} &= Z_P^0 + S\partial_{v_2} + 2R\partial_{v_3} + (4y'S_{y'} - 12S)\partial_{\bar{u}_{13}} + (2S - y'S_{y'})\partial_{\bar{u}_{14}} + (4R - 2y'R_{y'})\partial_{\bar{u}_{15}},\\
\bar{X}_{23} &= -2\partial_{v_3} + y'S(2\partial_{\bar{u}_{14}} - 12\partial_{\bar{u}_{13}}) - (2 + 4y'R)\partial_{\bar{u}_{15}},\\
\bar{X}_{24} &= -2y'\partial_{\bar{u}_{15}},\\
X_{26} &= \partial_{Q_x} + \partial_{P_{y'}} + (2S_x - 6R_{y'})\partial_{\bar{u}_{13}} + (R_{y'} - S_x - \alpha)\partial_{\bar{u}_{14}} + 2(2Q_{y'} - R_x - \beta)\partial_{\bar{u}_{15}},
\end{aligned}
$$

$$\bar{X}_{27} = Z_Q^0 + 2S\partial_{v_1} - R\partial_{v_2} - 4Q\partial_{v_3} + (12R - 6y'R_{y'})\partial_{\bar{u}_{13}} + y'(R_{y'} - \alpha)\partial_{\bar{u}_{14}} + (4y'Q_{y'} - 2y'\beta - 4Q)\partial_{\bar{u}_{15}},$$
$$\bar{X}_{28} = -2\partial_{v_2} + (4 + 10y'R)\partial_{\bar{u}_{13}} - \partial_{\bar{u}_{14}} - 4y'Q\partial_{\bar{u}_{15}},$$
$$\bar{X}_{29} = y'\big(6\partial_{\bar{u}_{13}} - \partial_{\bar{u}_{14}}\big),$$
$$X_{30} = \partial_{R_x} + \partial_{Q_{y'}} + 6\beta\partial_{\bar{u}_{13}} + (R_x + Q_{y'})\partial_{\bar{u}_{14}} + 2(2Q_x - P_{y'} + \gamma)\partial_{\bar{u}_{15}},$$
$$\bar{X}_{31} = Z_R^0 - 4R\partial_{v_1} - Q\partial_{v_2} + 2P\partial_{v_3} + 6y'\beta\partial_{\bar{u}_{13}} + (y'Q_{y'} + y'QR - 2Q)\partial_{\bar{u}_{14}} + 2y'(\gamma - P_{y'})\partial_{\bar{u}_{15}},$$
$$\bar{X}_{32} = -2\partial_{v_1} + y'Q\big(8\partial_{\bar{u}_{13}} - 2\partial_{\bar{u}_{14}}\big),$$
$$X_{33} = \partial_{S_x} + \partial_{R_{y'}} + (2P_{y'} - 6Q_x - 6\gamma)\partial_{\bar{u}_{13}} + (Q_x - P_{y'} + \gamma)\partial_{\bar{u}_{14}} - 2P_x\partial_{\bar{u}_{15}},$$
$$\bar{X}_{34} = Z_S^0 + 2Q\partial_{v_1} + P\partial_{v_2} + y'(2P_{y'} - 6\gamma)\partial_{\bar{u}_{13}} + y'(\gamma - P_{y'})\partial_{\bar{u}_{14}},$$
$$X_{35} = \partial_{S_{y'}}.$$

These operators are the coefficients of the fourth-order derivatives of w in (15). The operator $\bar{X}_{25}$ is a linear function of other operators of system (16) (see appendix A). Therefore, in the space of variables (18) the system $X_{21}I = 0, \ldots, X_{35}I = 0$ has 27 independent solutions

$$y', P, Q, R, S, \alpha, \beta, \gamma, A_1, A_2, B_1, B_2, \Gamma_1, \Gamma_2, M_1, \ldots, M_8,$$
$$L_1, L_2, L_3, L_4, M_0 = B_{2y'} + Q(A_2 + B_1) - RB_2 - S\Gamma_2 + \beta_y/2 + (\beta^2 - \alpha\gamma)/4, \qquad (19)$$

where α, β, γ are given by formulas (7).

In variables (19) next ten operators of system (16) take the form

$$X_{11} = \partial_P, \quad \bar{X}_{12} = 2\partial_\gamma, \quad \bar{X}_{13} = \partial_{B_2} - 2\partial_{\Gamma_1}, \quad \bar{X}_{14} = -1/2\partial_{M_0}, \quad X_{15} = \partial_Q,$$
$$\bar{X}_{16} = 2\partial_\beta, \quad \bar{X}_{17} = 2\partial_{A_2} - \partial_{B_1}, \quad X_{18} = \partial_R, \quad \bar{X}_{19} = 2\partial_\alpha, \quad X_{20} = \partial_S.$$

In the space of variables (19) operators $X_{11}, \ldots, X_{20}$ possess 17 independent invariants

$$y', A_1, A_0, B_0, \Gamma_2, M_1, \ldots, M_8, L_1, L_2, L_3, L_4. \qquad (20)$$

In these variables, the remaining eight operators $X_2, X_3, X_5, \ldots, X_{10}$ of system (16) become

$$X_2 = \partial_{y'},$$
$$X_3 = y'\partial_{y'} - 3A_1\partial_{A_1} - 2A_0\partial_{A_0} - B_0\partial_{B_0} - 4M_1\partial_{M_1} - 3M_2\partial_{M_2} - 3M_3\partial_{M_3} - 2M_4\partial_{M_4} - 2M_5\partial_{M_5} - M_6\partial_{M_6} - M_7\partial_{M_7} - 4L_1\partial_{L_1} - 3L_2\partial_{L_2} - 2L_3\partial_{L_3} - L_4\partial_{L_4},$$
$$X_5 = -3A_1\partial_{A_0} - 2A_0\partial_{B_0} - B_0\partial_{\Gamma_2} - M_1\big(\partial_{M_2} + 3\partial_{M_3}\big) - (3M_2 + M_3)\partial_{M_4} - 2M_3\partial_{M_5} - (2M_4 + M_5)\partial_{M_6} - M_5\partial_{M_7} - (M_6 + M_7)\partial_{M_8} - 3L_1\partial_{L_2} - 2L_2\partial_{L_3} - L_3\partial_{L_4},$$
$$\bar{X}_6 = 3A_1\partial_{M_3} + A_0\big(\partial_{M_4} + 2\partial_{M_5}\big) + B_0(2\partial_{M_6} + \partial_{M_7}) + 3\Gamma_2\partial_{M_8} + M_1\partial_{L_1} + M_3\partial_{L_2} + M_5\partial_{L_3} + M_7\partial_{L_4},$$
$$\bar{X}_7 = -\tfrac{3}{2}\big(A_1\partial_{L_1} + A_0\partial_{L_2} + B_0\partial_{L_3} + \Gamma_2\partial_{L_4}\big),$$
$$X_8 = -3A_1\partial_{A_1} - A_0\partial_{A_0} + B_0\partial_{B_0} + 3\Gamma_2\partial_{\Gamma_2} - 4M_1\partial_{M_1} - 2M_2\partial_{M_2} - 2M_3\partial_{M_3} + 2M_6\partial_{M_6} + 2M_7\partial_{M_7} + 4M_8\partial_{M_8} - 3L_1\partial_{L_1} - L_2\partial_{L_2} + L_3\partial_{L_3} + 3L_4\partial_{L_4},$$
$$\bar{X}_9 = 3A_1\partial_{M_1} + A_0\big(\partial_{M_2} + 2\partial_{M_3}\big) + B_0\big(2\partial_{M_4} + \partial_{M_5}\big) + 3\Gamma_2\partial_{M_6} - M_2\partial_{L_1} - M_4\partial_{L_2} - M_6\partial_{L_3} - M_8\partial_{L_4},$$
$$X_{10} = A_0\partial_{A_1} + 2B_0\partial_{A_0} + 3\Gamma_2\partial_{B_0} + (M_2 + M_3)\partial_{M_1} + M_4\partial_{M_2} + (M_4 + 2M_5)\partial_{M_3} + 2M_6\partial_{M_4} + (M_6 + 3M_7)\partial_{M_5} + M_8\big(3\partial_{M_6} + \partial_{M_7}\big) + L_2\partial_{L_1} + 2L_3\partial_{L_2} + 3L_4\partial_{L_3}.$$

In the space of variables (20) the operators $X_2, \bar{X}_6, \bar{X}_7, \bar{X}_9$ have 13 invariants

$$A_1, A_0, B_0, \Gamma_2, j_0, j_1, J_2, j_3, \ldots, j_8 \qquad (21)$$

defined by (8), (10), (11). In variables (21) the remaining operators take the form

$$\begin{aligned}
X_5 &= -3A_1\partial_{A_0} - 2A_0\partial_{B_0} - B_0\partial_{\Gamma_2} - j_0\partial_{j_1} - 5j_3\partial_{j_4} - 4j_4\partial_{j_5} - 3j_5\partial_{j_6} - 2j_6\partial_{j_7} - j_7\partial_{j_8},\\
X_{10} &= A_0\partial_{A_1} + 2B_0\partial_{A_0} + 3\Gamma_2\partial_{B_0} + j_1\partial_{j_0} + j_4\partial_{j_3} + 2j_5\partial_{j_4} + 3j_6\partial_{j_5} + 4j_7\partial_{j_6} + 5j_8\partial_{j_7},\\
X_8 &= -3A_1\partial_{A_1} - A_0\partial_{A_0} + B_0\partial_{B_0} + 3\Gamma_2\partial_{\Gamma_2} - j_0\partial_{j_0} + j_1\partial_{j_1} - 5j_3\partial_{j_3} - 3j_4\partial_{j_4}\\
&\quad - j_5\partial_{j_5} + j_6\partial_{j_6} + 3j_7\partial_{j_7} + 5j_8\partial_{j_8},\\
X_3 &= -3A_1\partial_{A_1} - 2A_0\partial_{A_0} - B_0\partial_{B_0} - 6j_0\partial_{j_0} - 5j_1\partial_{j_1} - 4J_2\partial_{J_2} - 6j_3\partial_{j_3}\\
&\quad - 5j_4\partial_{j_4} - 4j_5\partial_{j_5} - 3j_6\partial_{j_6} - 2j_7\partial_{j_7} - j_8\partial_{j_8}.
\end{aligned}$$

In the space of variables (21) the system $X_5I = 0$, $X_8I = 0$, $X_{10}I = 0$ has 10 functionally independent solutions (11). The last operator X_3 becomes

$$\begin{aligned}
X_3 &= -6J_0\partial_{J_0} - 5J_1\partial_{J_1} - 4J_2\partial_{J_2} - 8J_3\partial_{J_3} - 10J_4\partial_{J_4}\\
&\quad - 14J_5\partial_{J_5} - 10J_6\partial_{J_6} - 15J_7\partial_{J_7} - 18J_8\partial_{J_8} - 12J_9\partial_{J_9}.
\end{aligned}$$

Its invariants (12) provide the basis of functionally independent solutions of system (16).

One can readily verify that the operators $X_3, X_5, \bar{X}_6, X_8, \bar{X}_9, X_{10}$ leave invariant two systems of equations of the second order

$$A_1, A_0, B_0, \Gamma_2 = 0; \qquad a, b, c = 0$$

and five systems of equations of the third order

$$j_0, j_1 = 0; \quad j_3, \ldots, j_8 = 0; \quad i_1, i_2, i_3, i_4, i_5 = 0; \quad k_1, k_2 = 0; \quad k_3, k_4, J_0 = 0$$

(that is why variables (10) and (11) are called the relative invariants). The invariance condition of these systems under remaining operators from (16) holds identically. Each equation $J_i = 0, i = 0, \ldots, 9$ also is invariant under the operators $X_1, \ldots, X_{84}$.

According to [6] the coordinates f, g, h of the invariant differentiation operator

$$\mathcal{D} = fD_x + gD_y + hD_{y'}$$

are found from the relations

$$\widetilde{X}f = f\xi_x + g\xi_y + h\xi_{y'}, \qquad \widetilde{X}g = f\eta_x + g\eta_y + h\eta_{y'}, \qquad \widetilde{X}h = f\zeta_x + g\zeta_y + h\zeta_{y'}, \quad (22)$$

where $\xi = -w_{y'}, \eta = w - y'w_{y'}, \zeta = w_x + y'w_y$ are the coordinates of generator (5) at the independent variables x, y, y'. Equalities (22) should be split by w and its derivatives $w_x, \ldots, \partial^6 w/\partial y'^6$. For the functions $f, \bar{g} = g - y'f, h$ this yields the system of linear first-order PDEs, where eight equations are nonhomogeneous:

$$\begin{aligned}
&X_8 f = -f, \qquad X_{10} f = -h, \qquad \bar{X}_9 f = X_3 \bar{g} = \bar{g},\\
&X_3 h = X_8 h = h, \qquad X_5 h = f, \qquad \bar{X}_6 h = -\bar{g}.
\end{aligned}$$

The remaining equations of the system are homogeneous, whence it follows that $f, \bar{g}, h$ depend only on variables (20). Taking into account that the operators $X_3, X_5, \bar{X}_6, X_8, \bar{X}_9, X_{10}$ act on the variables $J_0, j_0, j_1, k_1, k_2, k_3, k_4$ as

$$\begin{aligned}
X_3 &= -6J_0\partial_{J_0} - 6j_0\partial_{j_0} - 5j_1\partial_{j_1} - 7k_1\partial_{k_1} - 6k_2\partial_{k_2} - 7k_3\partial_{k_3} - 6k_4\partial_{k_4},\\
X_5 &= -j_0\partial_{j_1} - k_1\partial_{k_2} - k_3\partial_{k_4}, \qquad X_{10} = j_1\partial_{j_0} + k_2\partial_{k_1} + k_4\partial_{k_3}, \qquad \bar{X}_6 = 7J_0\partial_{k_4},\\
X_8 &= -j_0\partial_{j_0} + j_1\partial_{j_1} - k_1\partial_{k_1} + k_2\partial_{k_2} - k_3\partial_{k_3} + k_4\partial_{k_4}, \qquad \bar{X}_9 = 7J_0\partial_{k_3},
\end{aligned}$$

one can find the general solution of this system

$$\begin{aligned}
f &= j_0 J_0^{-1} F(I) + k_1 J_0^{-7/6} H(I) + k_3 J_0^{-7/6} G(I), \qquad g = y'f + 7J_0^{-1/6} G(I),\\
h &= -j_1 J_0^{-1} F(I) - k_2 J_0^{-7/6} H(I) - k_4 J_0^{-7/6} G(I),
\end{aligned}$$

where $F(I)$, $G(I)$ and $H(I)$ are arbitrary functions of variables (12). Equating (F, G, H) to $(1, 0, 0)$, $(0, 0, 1)$ and $(0, 1, 0)$ respectively, one obtains three independent operators of invariant differentiation (13).

The last part of the proof is rather standard. It is not difficult to show that the invariant differentiations of (12) yield 24 independent fourth-order invariants. On the other hand, one can obtain the fourth-order invariants from the invariance criterion $\widetilde{X}I = 0$. Extension of generator (5) to the fourth-order derivatives of P, Q, R, S and splitting the equality $\widetilde{X}I = 0$ by $w, w_x, \ldots, \partial^7 w/\partial y'^7$ give rise to the system $X_1 I = 0, \ldots, X_{120} I = 0$ with 10 equations being again the linear combinations of the remaining 110 equations. Hence, in the 143-dimensional space $R_{143}(x, y, y', P, Q, R, S, P_x, \ldots, S_{y'y'y'y'})$ one obtains 33 independent invariants of equation (1). Nine of them are of the third order and 24 are of the fourth order, which coincides with the number of invariants obtained from (12) by invariant differentiations (13). Similar reasoning extended to the higher orders implies that invariants (12) form the basis. This completes the proof.

3. Examples of equivalent equations

Example 1. Let us consider the equation

$$\bar{y}''' = \frac{\bar{x}^5}{2}(\bar{y}'^2 + k)\bar{y}''^3 - \frac{5}{2}\frac{\bar{y}''}{\bar{x}}, \qquad k = \text{const}, \tag{23}$$

with the invariants $I_1 = -36\sqrt{5\theta}E^{-5/6}$, where $\theta = \bar{y}'^2\bar{x}^{-4}(\bar{y}'^2 + k)^{-3}$, $E = 3\theta + 1$,

$$\begin{aligned}
&I_2 = \tfrac{1}{3}E^{-2/3}[3\theta\bar{y}'^{-2}(\bar{y}'^2 + k) - 79], \qquad I_3 = 20E^{-4/3}[3\theta\bar{y}'^{-2}(2\bar{y}'^2 + k) - 2],\\
&I_4 = 180E^{-5/3}[5\theta^2\bar{y}'^{-4}(\bar{y}'^2 + k)(19\bar{y}'^2 - k) + 2\theta\bar{y}'^{-2}(711\bar{y}'^2 + 5k) + 495],\\
&I_5 = \tfrac{15}{4}E^{-7/3}[\theta^2\bar{y}'^{-4}(184\bar{y}'^2 + k)(k - 212\bar{y}'^2) + 12\theta\bar{y}'^{-2}(1523\bar{y}'^2 - 31k) + 34\,596],\\
&I_6 = 180E^{-5/3}[\theta^2\bar{y}'^{-4}(\bar{y}'^2 + k)(3\bar{y}'^2 + k) + 2\theta\bar{y}'^{-2}(141\bar{y}'^2 - 20k) + 151], \ldots
\end{aligned} \tag{24}$$

and find if the ODE (23) may be equivalent to the equation

$$y''' = \frac{y''^2}{2y'} - \frac{B'}{2B}y'' + \frac{y'}{32B^2}(A + 5B'^2 - 8BB'') + \frac{f(y)}{By'}, \qquad f(y) = K_0 + K_1 y + K_2 y^2, \tag{25}$$

where $K_0, K_1, K_2 = \text{const}$, $A = A(x)$, $B = B(x) \not\equiv 0$. It is known that equation (25) linearizes on differentiation [13]. The first two invariants of the ODE (25) read as

$$\begin{aligned}
&I_1 = -14400\epsilon^{-5/6}By'^2[2ABf'y'^2 + 3hfy' + 640B^2ff'],\\
&I_2 = \tfrac{1}{3}\epsilon^{-2/3}[(675B'h - 300Bh' - 19A^2 + 38\,400K_2B^3)y'^4 + 5600ABfy'^2 - 2022\,400B^2f^2],
\end{aligned}$$

where $h = 3AB' - 2BA'$, $\epsilon = 15(320B^2f' + hy')^2y'^4 + (160Bf - Ay'^2)^3$. Note that only one of the invariants (24), namely $I_1 \not\equiv 0$, depend on a single variable θ. Equation (25) possesses the same property when $A(x) = 0$, $K_2 \neq 0$, $K_3 \equiv K_1^2 - 4K_0K_2 \neq 0$. In this case, its invariants become $I_1 = -9\sqrt{10\tau}\varepsilon^{-5/6}$, where $\tau = Bf'^2f^{-3}y'^4$, $\varepsilon = (3\tau/8 + 1)$,

$$\begin{aligned}
&I_2 = \tfrac{1}{6}\varepsilon^{-2/3}[3\tau K_2 f/f'^2 - 158], \qquad I_3 = \tfrac{5}{2}\varepsilon^{-4/3}[3\tau(8K_2 f + K_3)/f'^2 - 16],\\
&I_4 = 45\varepsilon^{-5/3}[5\tau^2 K_2 f(19K_2 f + 5K_3)/f'^4 + 2\tau(1422K_2 f + 353K_3)/f'^2 + 1980],\\
&I_5 = \tfrac{15}{256}\varepsilon^{-7/3}[-\tau^2(736K_2 f + 183K_3)(848K_2 f + 213K_3)/f'^4\\
&\qquad - 192\tau(3046K_2 f + 777K_3)/f'^2 + 2214\,144],\\
&I_6 = \tfrac{45}{2}\varepsilon^{-5/3}[\tau^2 K_2 f(6K_2 f + K_3)/f'^4 + 2\tau(564K_2 f + 161K_3)/f'^2 + 1208], \ldots.
\end{aligned}$$

Equating them to invariants (24) and taking into account that $4K_2 f = f'^2 - K_3$, we find $\tau = 8\theta$, $\bar{y}' = C(y + K_1/2K_2)$, $k = -C^2K_3/4K_2^2$, $\bar{x} = y'^{-1}(2K_2/B)^{1/4}/C$, $C =$ const. For the sake of simplicity, we set $C = 1$, $K_2 = 1/2$, $K_1 = 0$, $K_0 = k/2$, $B(x) = 1$. From relations (3), one can readily obtain the missing function $\Psi(x, y, y')$ in transformation (2) and then establish that the contact change of variables

$$\bar{x} = \frac{1}{y'}, \qquad \bar{y} = \frac{y}{y'} - x, \qquad \bar{y}' = y \tag{26}$$

turns the ODE (23) into the equation

$$y''' = \frac{y''^2 + y^2 + k}{2y'}.$$

It linearizes on differentiation, taking the form $y^{IV} - y = 0$, with the general solution

$$y = c_0\,\mathrm{e}^x + c_1\,\mathrm{e}^{-x} + c_2 \sin x + c_3 \cos x,$$

where the constants c_i satisfy the relation $4c_0c_1 + c_2^2 + c_3^2 + k/2 = 0$. The substitution of this function into (26) leads to the solution of the ODE (23) in a parametric form

$$\bar{x} = \frac{1}{c_0\,\mathrm{e}^x - c_1\,\mathrm{e}^{-x} + c_2 \cos x - c_3 \sin x}, \qquad \bar{y} = \frac{c_0\,\mathrm{e}^x + c_1\,\mathrm{e}^{-x} + c_2 \sin x + c_3 \cos x}{c_0\,\mathrm{e}^x - c_1\,\mathrm{e}^{-x} + c_2 \cos x - c_3 \sin x} - x$$

with the parameter x. This example shows how the use of invariants enables one to obtain the solution of an equation under study via the solution of an equivalent equation.

Example 2. The third-order ODE

$$\bar{y}''' = \frac{4m\bar{y}''^2 - \bar{y}''}{2(\bar{x} - 2m\bar{y}')}, \qquad m = \text{const} \tag{27}$$

has the constant invariants

$$\begin{aligned} &I_1 = 0, \quad I_2 = -79/3, \quad I_3 = -40, \quad I_4 = 89\,100, \quad I_5 = 129\,735,\\ &I_6 = 27\,180, \quad I_7 = 0, \quad I_8 = 4021\,785, \quad I_9 = -47\,400. \end{aligned} \tag{28}$$

Equation (25), which linearizes on differentiation [13], has the constant invariants in two cases. When $A(x) = B^{3/2}$, $f(y) = 0$ it has the invariants

$$\begin{aligned} &I_1 = 0, \quad I_2 = -19/3, \quad I_3 = 80, \quad I_4 = 2700, \quad I_5 = 540,\\ &I_6 = -900, \quad I_7 = 0, \quad I_8 = -1080, \quad I_9 = 2280. \end{aligned}$$

When $A(x) = 0$, $f(y) = K_0$, $K_0 \neq 0$, invariants of the ODE (25) coincide with (28) and hence, it is equivalent to equation (27). Direct calculation shows that the contact change of variables

$$\bar{x} = 2mx + y', \qquad \bar{y} = xy' - y + mx^2, \qquad \bar{y}' = x \tag{29}$$

transforms the ODE (27) to equation (25) with $B(x) = 1$, $K_0 = -2m^2$

$$y''' = \frac{y''^2}{2y'} - 2\frac{m^2}{y'}, \tag{30}$$

which takes the form $y^{IV} = 0$ after a differentiation. The general solution of the ODE (30) is given by the formula

$$y = c_0 + c_1x + c_2x^2 + c_3x^3,$$

where the constants c_i satisfy a single quadratic relation $3c_1c_3 - c_2^2 + m^2 = 0$. Substituting it into (29) and then eliminating x from

$$\bar{x} = 3c_3x^2 + 2(c_2 + m)x + c_1, \qquad \bar{y} = 2c_3x^3 + (c_2 + m)x^2 - c_0,$$

one arrives at the general solution of the ODE (27)

$$\bar{y} = \bar{c}_0 + \bar{c}_1\bar{x} + \bar{c}_2(\bar{x} - 2m\bar{c}_1)^{3/2},$$

where $\bar{c}_0 = \bar{c}_1^2(c_2 - 5m)/3 - c_0$, $\bar{c}_1 = -(c_2 + m)/(3c_3)$, $\bar{c}_2 = 2/\sqrt{27c_3}$.

Example 3. In [14], the higher order analogies to the Painlevé equations are studied. In an autonomous case one of them reads as

$$v^{IV} = 4\frac{v'v'''}{v} + 3\frac{v''^2}{v} - \frac{21}{2}\frac{v'^2v''}{v^2} + \frac{9}{2}\frac{v'^4}{v^3} + 10\delta\left(\frac{2v'^2}{v^3} - \frac{v''}{v^2}\right) - K_0 + K_2v^2 + 8\frac{\delta^2}{v^3}. \tag{31}$$

Another fourth-order ODE presented in [14] has the form

$$v^{IV} = 3\frac{v'v'''}{v} + \frac{7}{2}\frac{v''^2}{v} - \frac{17}{2}\frac{v'^2v''}{v^2} + \frac{27}{8}\frac{v'^4}{v^3} + K_3\left(2v'' - \frac{v'^2}{v}\right)$$
$$+5\delta\left(\frac{3v'^2}{v^3} - \frac{2v''}{v^2}\right) + 2K_2v^2 - 4K_3\frac{\delta}{v} + 6\frac{\delta^2}{v^3}. \tag{32}$$

As is shown in [15], when $K_3 = 0$, the third-order ODE

$$v''' = \frac{v''^2}{2v'} + 2\frac{v'v''}{v} - \frac{9}{8}\frac{v'^3}{v^2} + K_3v' - 5\delta\frac{v'}{v^2} + \frac{F(v)}{v'},$$
$$F(v) = K_0v + K_1v^2 + K_2v^3 + 4K_3\delta - 2\frac{\delta^2}{v^2} \tag{33}$$

represents an integral of equation (31). If $K_1 = 0$, equation (33) is an integral of the ODE (32). The invariants of the ODE (33) are calculated by formulas (12), but the result is too cumbersome. So, only the first two invariants are given here:

$$I_1 = 7200E^{-5/6}v'^2[15\delta v'^6 + v'^4(8\delta H + 15G) - 4v'^2(GH + 70\delta v^2F) - 40v^2FG],$$
$$I_2 = -\tfrac{1}{3}E^{-2/3}[4275v'^8 + 120v'^6(135\delta - 11K_3v^2) - 11\,200v^2v'^2FH + 126\,400v^4F^2$$
$$+ 16v'^4(76K_3^2v^4 - 1960K_3\delta v^2 + 4300\delta^2 - 75(K_0v^3 + K_2v^5) - 1725v^2F)], \tag{34}$$

where $G = K_2v^5 - K_0v^3 - 8K_3\delta v^2 + 8\delta^2$, $H = K_3v^2 - 5\delta$,

$$E = 3375v'^{12} - 5400v'^{10}H + 120v'^8\left(225v^2F + 24K_3^2v^4 - 240K_3\delta v^2 + 3800\delta^2\right)$$
$$+ 128v'^6(1500\delta G - 225v^2FH - 4H^3) + 960v'^4(75v^4F^2 + 8v^2FH^2 + 25G^2)$$
$$- 38\,400v^4v'^2F^2H + 64\,000v^6F^3.$$

We suppose that equation (33) may be equivalent to an equation, which follows from the system with the Hénon–Heiles Hamiltonian

$$H = \frac{1}{2}(p_1^2 + p_2^2 + \omega_1q_1^2 + \omega_2q_2^2) + Aq_1q_2^2 - \frac{B}{3}q_1^3 + \frac{\lambda}{2}q_2^{-2}, \qquad \omega_1, \omega_2, A, B, \lambda = \text{const}.$$

All its integrable cases, when Hamiltonian's equations of motion

$$q_1'' + \omega_1q_1 - Bq_1^2 + Aq_2^2 = 0, \qquad q_2'' + \omega_2q_2 + 2Aq_1q_2 - \lambda q_2^{-3} = 0 \tag{35}$$

are solved in terms of hyperelliptic functions of genus two, are presented in [16]. One of them, namely the KdV5 case, corresponds to $B = -6A$. Then elimination of q_1 between two equations (35) gives rise to the equation

$$y^{IV} = 2\frac{y'y'''}{y} + 4\frac{y''^2}{y} - 2\frac{y'^2y''}{y^2} + 2k_3y'' + 10\lambda\left(\frac{2y'^2}{y^5} - \frac{y''}{y^4}\right) + 2k_1y + 4k_2y^3 - 2k_3\frac{\lambda}{y^3} + 3\frac{\lambda^2}{y^7}$$

for the function $y = q_2$, with the parameters $k_1 = \omega_2(3\omega_2 - \omega_1)/2$, $k_2 = A^2/2$, $k_3 = 3\omega_2 - \omega_1/2$. This fourth-order ODE possesses an integral

$$y''' = \frac{y''^2}{2y'} + 2\frac{y'y''}{y} + k_3y' - 5\lambda\frac{y'}{y^4} + \frac{f(y)}{y'}, \qquad f(y) = k_0 + k_1y^2 + k_2y^4 + k_3\frac{\lambda}{y^2} - \frac{\lambda^2}{2y^6}. \tag{36}$$

The third-order equation (36) has the invariants

$$I_1 = 1800\varepsilon^{-5/6}y^2y'^2[15\lambda y^6y'^6 + y^4y'^4(2\lambda h + 15g) - y^2y'^2(gh + 70\lambda y^6 f) - 10y^6fg],$$
$$I_2 = -\tfrac{1}{3}\varepsilon^{-2/3}\big[4275y^8y'^8 + 30y^6y'^6(135\lambda - 11k_3y^4) - 700y^8y'^2fh + 7900y^{12}f^2 + y^4y'^4\big(76k_3^2y^8 - 1960k_3\lambda y^4 + 4300\lambda^2 - 300(k_0y^6 + k_2y^{10}) - 6900y^6f\big)\big], \tag{37}$$

where

$$g = k_2y^{10} - k_0y^6 - 2k_3\lambda y^4 + 2\lambda^2, \qquad h = k_3y^4 - 5\lambda,$$
$$\varepsilon = 3375y^{12}y'^{12} - 1350y^{10}y'^{10}h + 30y^8y'^8(225y^6f + 6k_3^2y^8 - 60k_3\lambda y^4 + 950\lambda^2) + 8y^6y'^6(1500\lambda g - 225y^6fh - h^3) + 60y^4y'^4(75y^{12}f^2 + 2y^6fh^2 + 25g^2) - 600y^8y'^2f^2h + 1000y^{18}f^3.$$

Equations (33) and (36) are both quadratic, but not cubic in v'' and y''. As follows from [11], an ODE (1) with $S(x, y, y') = 0$ and an ODE (4) with $\bar{S}(\bar{x}, \bar{y}, \bar{y}') = 0$ may be connected either by point transformation $\bar{x} = \phi(x, y), \bar{y} = \psi(x, y)$, or by contact transformation (2) such that $\Phi(x, y, y') = \text{const}$ is an integral of the ODE (1) (see, e.g., ODEs (30), (41) and transformations (29) and (42) respectively).

Invariants (34) and (37) do not involve the independent variable. Comparing them we can suppose that the independent variable remains unaltered and equations (33) and (36) are connected by the point transformation of the form $v = y^n$. Indeed, the change of variables

$$v = y^2, \qquad v' = 2yy'$$

turns the ODE (33) into equation (36) with the parameters $4k_0 = K_0, 4k_1 = K_1, 4k_2 = K_2, k_3 = K_3, \lambda = \delta$. This relation between integrals (33) and (36) of the fourth-order ODEs was not found either in [14] or in [15]. Thus, we can conclude that in an autonomous case, the fourth-order analogies to the Painlevé equations given by ODEs (31) and (32) are related to an integrable case of Hénon–Heiles Hamiltonian and hence, they are solved in terms of hyperelliptic functions of genus two.

Example 4. The Caudrey–Dodd–Gibbon equation [17]

$$u_t = u_{xxxxx} - 30uu_{xxx} - 30u_xu_{xx} + 180u^2u_x \tag{38}$$

possesses an invariant manifold defined by the third-order ODE

$$u_{xxx} = \frac{2}{3}\frac{u_{xx}^2}{u_x} + 2(u^2 - s)\frac{u_{xx}}{u_x} + 6uu_x - \frac{12}{u_x}(u^2 - s)^2, \qquad s = \text{const} \tag{39}$$

(new infinite sequence of invariant manifolds of equation (38) see in [18]). Application of invariant manifolds generalizes the separation of variables method for evolution equations: first we integrate the ODE (39) with respect to x and then substitute its solution $u(t, x) = f(x, c_1(t), c_2(t), c_3(t))$ into equation (38). This yields the system of first-order ODEs for the functions $c_i(t)$, which arose as constants of integration in solving the ODE (39).

Equation (39) has invariants

$$I_1 = 24\sqrt{30}E^{-5/6}u_x^2[7uu_x^4 + 6(u^2 - s)(109u^2 - 45s)u_x^2 - 2304u(u^2 - s)^3],$$
$$I_2 = -\tfrac{4}{45}E^{-2/3}[5(137u^2 - 21s)u_x^4 + 672u(u^2 - s)^2u_x^2 + 42\,480(u^2 - s)^4],$$

where $E = 2\big[49u_x^8 + 224u(9s - 13u^2)u_x^6 + 576(u^2 - s)^2(13u^2 + 3s)u_x^4 - 13\,824u(u^2 - s)^4u_x^2 - 55\,296(u^2 - s)^6\big]$. When $s \equiv 0$, it is not difficult to see that these invariants (as well as other invariants of the ODE (39) omitted here) may be represented as the functions of a single variable $\bar{x} = u^{3/2}/u_x$:

$$I_1 = 12\sqrt{15}\varepsilon^{-5/6}(2\bar{x})^{2/3}(7 + 654\bar{x}^2 - 2304\bar{x}^4),$$
$$I_2 = -\tfrac{1}{3}\varepsilon^{-2/3}\bar{x}^{4/3}(685 + 672\bar{x}^2 + 42\,480\bar{x}^4), \tag{40}$$

where $\varepsilon = 49 - 2912\bar{x}^2 + 7488\bar{x}^4 - 13\,824\bar{x}^6 - 55\,296\bar{x}^8$. Invariants (40) depend on one variable, which is not sufficient to find an equation equivalent to the ODE (39). However, we can suppose that the ODE (39) with $s = 0$ is equivalent to an equation with the invariants depending on $\bar{x}$ only, for example, to one of the ODEs

$$\bar{u}''' = a_3(\bar{x})\bar{u}''^3 + a_2(\bar{x})\bar{u}''^2 + a_1(\bar{x})\bar{u}'' + a_0(\bar{x}),$$

$$\bar{u}''' = a_3(\bar{x})\frac{\bar{u}''^3}{\bar{u}'^2} + (a_2(\bar{x}) + 1)\frac{\bar{u}''^2}{\bar{u}'} + a_1(\bar{x})\bar{u}'' + a_0(\bar{x})\bar{u}'.$$

The substitution $\bar{u}'' = w$ reduces the first ODE to solving the Abel equation

$$w' = a_3(\bar{x})w^3 + a_2(\bar{x})w^2 + a_1(\bar{x})w + a_0(\bar{x})$$

and two quadratures. The second ODE is reduced to the Abel equation by the substitution $\bar{u}''/\bar{u}' = w$. Invariants (40) provide only one function in transformation (2), namely $\Phi = u^{3/2}/u_x$. The remaining part of transformation (2) is found from equations (3). By this way, one obtains the contact transformation

$$\bar{x} = \frac{u^{3/2}}{u_x}, \qquad \bar{u} = x + 2\frac{u}{u_x}, \qquad \bar{u}' = \frac{2}{\sqrt{u}}$$

connecting the ODE (39) with $s = 0$ and the equation

$$\bar{u}''' = 6\bar{x}(4\bar{x}^2 - 1)(1 - 2\bar{x}^2)\frac{\bar{u}''^3}{\bar{u}'^2} + (4\bar{x}^2 - 5)\frac{\bar{u}''^2}{\bar{u}'} - \frac{7}{3}\frac{\bar{u}''}{\bar{x}}.$$

Example 5. Another example of this kind of equations provides an ODE

$$y''' = \frac{ly''[2xy'' - (l+1)y']}{(l+1)[2xy' - (l+2)y]}, \qquad l = \text{const}, \tag{41}$$

which arises in the classification of integrable Hamiltonian hydrodynamic chains associated with Kupershmidt's brackets [19]. It possesses a remarkable property: for parameter values $l = 1, 2, 3, \ldots$, equation (41) linearizes on exactly l differentiations [19]. Also the family of equations (41) admits the contact equivalence transformation

$$\tilde{x} = x - (l+2)\frac{y}{y'}, \qquad \tilde{y} = yy'^{-\frac{l+2}{l+1}}, \qquad \tilde{y}' = -\frac{1}{l+1}y'^{-\frac{1}{l+1}} \tag{42}$$

mapping the ODE (41) into an equation of the same form with the parameter $\tilde{l} = -l/(l+1)$.

It is readily seen that invariants

$$I_1 = 18L^2\sqrt{l(l^2+L^2)}E^{-5/6}(x^2y'^2 - Lxyy')^{-2/3}y(2xy' - Ly)^2, \qquad L = l+2,$$

$$\begin{aligned}I_2 = &-\tfrac{1}{12}E^{-2/3}(x^2y'^2 - Lxyy')^{-4/3}[9(l^2+3L^2)(2xy'-Ly)^4\\ &- 12(l^2+L^2)^{-1}(13l^4 + 51l^2L^2 + 18(l+L)^2)xy'(xy'-y)(2xy'-Ly)^2\\ &+ 16l^2(l^2+L^2)^{-2}(35l^4 - 279l^2L^2 - 339(l+L)^2)x^2y'^2(xy'-Ly)^2],\end{aligned}$$

$$\begin{aligned}I_3 = &\,4l^2E^{-4/3}(x^2y'^2 - Lxyy')^{-2/3}[40l^2(11l^2+21L^2)(l^2+L^2)^{-1}x^2y'^2(xy'-Ly)^2\\ &+ 240l^2L^2xy^2y'(xy'-Ly) - 27(l^2+L^2)(2xy'-Ly)^4],\end{aligned}$$

where $E = -4l(l+3)(2l+3)(2xy'-Ly)^2/5 - 2l^3L^2y^2$, and other invariants of equation (41) are the functions of a single variable $xy'y^{-1}$. This yields the function $\Phi(x, y, y')$ in transformation (2). The whole change of variables found from conditions (3),

$$\bar{x} = \frac{y}{xy'}, \qquad \bar{y} = \frac{y}{xy'}\ln y - \ln x, \qquad \bar{y}' = \ln y,$$

transforms the ODE (41) to the equation

$$\bar{y}''' = \bar{x}(2\bar{x}-1)(1-\bar{x})\bar{y}''^3 + \left((1-\bar{x})\bar{y}''^2 - \frac{\bar{y}''}{\bar{x}}\right)\left(\frac{l}{l+1}\left(\bar{x}\bar{y}'' + \frac{\bar{x}^2\bar{y}''+2}{(l+2)\bar{x}-2}\right)+3\right).$$

4. Concluding remarks

One of the main ways of solving nonlinear ODEs is to study if they are equivalent to a historically solved ODE. Theory of equivalence of differential equations deals with different problems. The first one is to establish if two given equations, say (1) and (4), are equivalent with respect to an invertible transformation (2). Whenever they are equivalent the second problem is to find transformation (2) connecting these equations. Since the equivalent equations should have equal invariants, both of these problems may be solved with the use of invariants of the family of equations (1) as follows. Using formulas (12), we calculate invariants $I_k(x, y, y'), k \geqslant 1$ for equation (1), invariants $\bar{I}_k(\bar{x}, \bar{y}, \bar{y}'), k \geqslant 1$ for equation (4) and then investigate whether the relations

$$I_1(x, y, y') = \bar{I}_1(\bar{x}, \bar{y}, \bar{y}'), \quad I_2(x, y, y') = \bar{I}_2(\bar{x}, \bar{y}, \bar{y}'), \quad I_3(x, y, y') = \bar{I}_3(\bar{x}, \bar{y}, \bar{y}'), \ldots \tag{43}$$

are consistent. There may be three possibilities as follows.

(1) In exceptional cases, when all invariants of two given ODEs (1) and (4) are constant and mutually equal, system (43) is obviously consistent. But transformation (2) connecting these equations can be found by direct computations only, i.e. by substituting (2) into (4), replacing y''' by virtue of (1), splitting the equality obtained by y'' and then integrating the resulting system of nonlinear PDEs for the functions Φ, Ψ, Ω and equations (3) together.

(2) Generally, as is seen from the previous section, invariants are not identically constant. Then from the first one or more algebraic equalities (43), taking into account (3), one can find a possible form of transformation (2). If substituting it into remaining relations (43) one obtains identities, then system (43) is consistent and equations (1) and (4) are equivalent under this transformation (2).

(3) When all invariants are the functions of a single combination of x, y, y', one can suppose that equation (1) is equivalent to an ODE with the invariants, which also depend on one variable only. One cannot find via the invariants the whole change of variables (2). Invariants provide only one function, say $\Phi(x, y, y')$, in transformation (2). Function $\Psi(x, y, y')$ is found as a solution of linear first-order PDE (3). It is simlper than seeking the transformation (2) directly, when one should integrate the system of nonlinear PDEs for the functions Φ, Ψ, Ω.

Note that invariants may be effective when we need to prove nonequivalence of two given equations. For example, if some invariant I_k of equation (1) is constant and the same invariant $\bar{I}_k$ of equation (4) is nonconstant, then these two equations are obviously nonequivalent. If an invariant I_k of equation (1) is a function of a single combination of variables x, y, y' and the same invariant $\bar{I}_k$ of equation (4) does not possess this property, then these two equations are nonequivalent too.

In this paper, we provide the basis of differential invariants (12) for the family of equations (1) as well as the operators of invariant differentiation (13). The result is given in terms of the functions (7)–(11) and is probably as compact as it could be already. Formulas (7)–(11) seem too large for calculations, but they are simple for programming in any symbolic package such as Maple or Mathematica.

Acknowledgments

This research was supported by grant MK-5863.2008.1 for young scientists, RFBR grants 09-01-00134, 09-01-92431-CE and Leading Scientific Schools grant 2215-2008-1.

Appendix A. Operators of the equivalence transformation group of the equation (1)

Here the operators $X_1, X_4, X_{21}, \ldots, X_{84}$ from system (16) are given. Note that 10 of these operators may be represented as linear functions of the 56 remaining operators. These are the operators $X_{41}, X_{61}, X_{62}, X_{63}, X_{68}, X_{69}, X_{74}$ and

$\bar{X}_{40} = -\bar{X}_{67}, \quad \bar{X}_{46} = -\bar{X}_{73}, \quad 2\bar{X}_{25} = P\bar{X}_{60} + Q\bar{X}_{67} + R\bar{X}_{73} + S\bar{X}_{78} - \bar{X}_{45} - \bar{X}_{72}.$

Each operator X_i is a coefficient of a certain derivative of w in the prolonged operator $\widetilde{X}$. Below we label once every operator X_i by the corresponding derivative of w. Instead of a part of operators X_i, we give their linear combinations $\bar{X}_i$, because it does not alter the solution of system (16).

We do not present here 18 operators

$$X_2(w_x), \quad X_3(w_y), \quad X_5(w_{xx}), \quad \bar{X}_6 = 2y'X_5 - X_6(w_{xy}),$$
$$\bar{X}_7 = y'^2X_5 - y'X_6 + X_7(w_{yy}), \quad X_8(w_{xy'}), \quad \bar{X}_9 = y'X_8 - X_9(w_{yy'}), \quad X_{10}(w_{y'y'}),$$
$$X_{11}(w_{xxx}), \quad \bar{X}_{12} = 3y'X_{11} - X_{12}(w_{xxy}), \quad \bar{X}_{13} = 3y'^2X_{11} - 2y'X_{12} + X_{13}(w_{xyy}),$$
$$\bar{X}_{14} = y'^3X_{11} - y'^2X_{12} + y'X_{13} - X_{14}(w_{yyy}), \quad X_{15}(w_{xxy'}),$$
$$\bar{X}_{16} = 2y'X_{15} - X_{15}(w_{xyy'}), \quad \bar{X}_{17} = y'^2X_{15} - y'X_{16} + X_{17}(w_{yyy'}),$$
$$X_{18}(w_{xy'y'}), \quad \bar{X}_{19} = y'X_{18} - X_{19}(w_{yy'y'}), \quad X_{20}(w_{y'y'y'}),$$

since they are too cumbersome. In appendix B, we provide the Maple program, which allows us to obtain all the coefficients $X_1, \ldots, X_{84}$ of the prolonged operator (15).

Using the notation $d = \partial_x + y'\partial_y$ and

$$Z_F^0 = y'\partial_{F_x} - \partial_{F_y}, \quad Z_F^1 = 2y'\partial_{F_{xx}} - \partial_{F_{xy}}, \quad Z_F^2 = y'^2\partial_{F_{xx}} - y'\partial_{F_{xy}} + \partial_{F_{yy}},$$
$$Z_F^3 = y'\partial_{F_{xy'}} - \partial_{F_{yy'}}, \quad Z_F^4 = 3y'\partial_{F_{xxx}} - \partial_{F_{xxy}}, \quad Z_F^5 = 3y'^2\partial_{F_{xxx}} - 2y'\partial_{F_{xxy}} + \partial_{F_{xyy}},$$
$$Z_F^6 = y'^3\partial_{F_{xxx}} - y'^2\partial_{F_{xxy}} + y'\partial_{F_{xyy}} - \partial_{F_{yyy}}, \quad Z_F^7 = 2y'\partial_{F_{xxy'}} - \partial_{F_{xyy'}},$$
$$Z_F^8 = y'^2\partial_{F_{xxy'}} - y'\partial_{F_{xyy'}} + \partial_{F_{yyy'}}, \quad Z_F^9 = y'\partial_{F_{xy'y'}} - \partial_{F_{yy'y'}}$$

with F being equal to P, Q, R or S, the operators $X_1, X_4, X_{21}, \ldots, X_{84}$ are as follows:

$$X_1(w) = \partial_y, \quad X_4(w_{y'}) = -\partial_x - y'\partial_y,$$
$$\begin{aligned} X_{21}(w_{xxxx}) = {} & \partial_{P_x} - 3Q\partial_{P_{xx}} - 2R\partial_{Q_{xx}} - S\partial_{R_{xx}} - (9Q_x + P_{y'})\partial_{P_{xxx}} - 3Q_y\partial_{P_{xxy}} \\ & - (6R_x + Q_{y'})\partial_{Q_{xxx}} - 2R_y\partial_{Q_{xxy}} - 3Q_{y'}\partial_{P_{xxy'}} - (3S_x + R_{y'})\partial_{R_{xxx}} - S_y\partial_{R_{xxy}} \\ & - 2R_{y'}\partial_{Q_{xxy'}} - S_{y'}\big(\partial_{S_{xxx}} + \partial_{R_{xxy'}}\big), \end{aligned}$$
$$\begin{aligned} \bar{X}_{22} = 4y'X_{21} - X_{22}(w_{xxxy}) = {} & Z_P^0 - 3QZ_P^1 - 2RZ_Q^1 - SZ_R^1 - \partial_{Q_{xx}} - 3\partial_{P_{xy'}} \\ & + 6Q_yZ_P^5 - (6\,\mathrm{d}Q + P_{y'})Z_P^4 + 4R_yZ_Q^5 - (4\,\mathrm{d}R + Q_{y'})Z_Q^4 - 3Q_{y'}Z_P^7 + 2S_yZ_R^5 \\ & - (2\,\mathrm{d}S + R_{y'})Z_R^4 - 2R_{y'}Z_Q^7 - S_{y'}\big(Z_S^4 + Z_R^7\big) - P\partial_{P_{xxx}} + 6Q\partial_{P_{xxy'}} \\ & + R\big(\partial_{R_{xxx}} + 4\partial_{Q_{xxy'}}\big) + 2S\big(\partial_{S_{xxx}} + \partial_{R_{xxy'}}\big), \end{aligned}$$
$$\begin{aligned} \bar{X}_{23} = 6y'^2X_{21} - 3y'X_{22} + X_{23}(w_{xxyy}) = {} & -3QZ_P^2 - 2RZ_Q^2 - SZ_R^2 - Z_Q^1 - 3Z_P^3 \\ & + 9Q_yZ_P^6 - (3\,\mathrm{d}Q + P_{y'})Z_P^5 + 6R_yZ_Q^6 - (2\,\mathrm{d}R + Q_{y'})Z_Q^5 - 3Q_{y'}Z_P^8 + 3S_yZ_R^6 \\ & - (\mathrm{d}S + R_{y'})Z_R^5 - 2R_{y'}Z_Q^8 - S_{y'}\big(Z_S^5 + Z_R^8\big) - PZ_P^4 + 6QZ_P^7 + R\big(Z_R^4 + 4Z_Q^7\big) \\ & + 2S\big(Z_S^4 + Z_R^7\big) + \partial_{Q_{xxy'}} + 6\partial_{P_{xy'y'}}, \end{aligned}$$
$$\begin{aligned} \bar{X}_{24} = 4y'^3X_{21} - 3y'^2X_{22} + 2y'X_{23} - X_{24}(w_{xyyy}) = {} & -Z_Q^2 - P_{y'}Z_P^6 - Q_{y'}Z_Q^6 - R_{y'}Z_R^6 \\ & - S_{y'}Z_S^6 - PZ_P^5 + 6QZ_P^8 + R\big(Z_R^5 + 4Z_Q^8\big) + 2S\big(Z_S^5 + Z_R^8\big) + Z_Q^7 + 6Z_P^9, \end{aligned}$$
$$\bar{X}_{25} = y'^4X_{21} - y'^3X_{22} + y'^2X_{23} - y'X_{24} + X_{25}(w_{yyyy}) = RZ_R^6 - PZ_P^6 + 2SZ_S^6 + Z_Q^8,$$

$$X_{26}(w_{xxxy'}) = \partial_{Q_x} + \partial_{P_{y'}} + 3P\partial_{P_{xx}} + Q\big(\partial_{Q_{xx}} - 3\partial_{P_{xy'}}\big) - R\big(\partial_{R_{xx}} + 2\partial_{Q_{xy'}}\big)$$
$$- S\big(3\partial_{S_{xx}} + \partial_{R_{xy'}}\big) + (10P_x + y'P_y)\partial_{P_{xxx}} + 3P_y\partial_{P_{xxy}} + (4Q_x + y'Q_y)\partial_{Q_{xxx}}$$
$$+ Q_y\big(\partial_{Q_{xxy}} - 3\partial_{P_{xyy'}}\big) + (2P_{y'} - 6Q_x)\partial_{P_{xxy'}} - 6Q_{y'}\partial_{P_{xy'y'}} + (y'R_y - 2R_x)\partial_{R_{xxx}}$$
$$- R_y\big(\partial_{R_{xxy}} + 2\partial_{Q_{xyy'}}\big) - 4R_x\partial_{Q_{xxy'}} - 4R_{y'}\partial_{Q_{xy'y'}} + (y'S_y - 8S_x)\partial_{S_{xxx}}$$
$$- S_y\big(3\partial_{S_{xxy}} + \partial_{R_{xyy'}}\big) - 2(S_x + R_{y'})\partial_{R_{xxy'}} - S_{y'}\big(4\partial_{S_{xxy'}} + 2\partial_{R_{xy'y'}}\big),$$

$$\bar{X}_{27} = 3y'X_{26} - X_{27}(w_{xxyy'}) = Z_Q^0 + 3PZ_P^1 + Q\big(Z_Q^1 - 3Z_P^3\big) - R\big(Z_R^1 + 2Z_Q^3\big)$$
$$- S\big(3Z_S^1 + Z_R^3\big) - \partial_{R_{xx}} - 3\partial_{Q_{xy'}} - 6\partial_{P_{y'y'}} - 6P_yZ_P^5 + 7\,\mathrm{d}PZ_P^4 + Q_y\big(6Z_P^8 - 2Z_Q^5\big)$$
$$+ 3\,\mathrm{d}QZ_Q^4 + (2P_{y'} - 3\,\mathrm{d}Q)Z_P^7 + R_y(2Z_R^5 + 4Z_Q^8) - \mathrm{d}R(Z_R^4 + 2Z_Q^7) - 6Q_{y'}Z_P^9$$
$$+ S_y\big(6Z_S^5 + 2Z_R^8\big) - 5\,\mathrm{d}SZ_S^4 - \big(\mathrm{d}S + 2R_{y'}\big)Z_R^7 - 4R_{y'}Z_Q^9 - S_{y'}\big(4Z_S^7 + 2Z_R^9\big)$$
$$- 4P\partial_{P_{xxy'}} + Q\big(12\partial_{P_{xy'y'}} - \partial_{Q_{xxy'}}\big) + R\big(2\partial_{R_{xxy'}} + 8\partial_{Q_{xy'y'}}\big) + S\big(5\partial_{S_{xxy'}} + 4\partial_{R_{xy'y'}}\big),$$

$$\bar{X}_{28} = 3y'^2X_{26} - 2y'X_{27} + X_{28}(w_{xyyy'}) = 3PZ_P^2 + QZ_Q^2 - RZ_R^2 - 3SZ_S^2 - Z_R^1 - 3Z_Q^3$$
$$- 9P_yZ_P^6 + 4\,\mathrm{d}PZ_P^5 - 3Q_yZ_Q^6 + 2\,\mathrm{d}QZ_Q^5 + 2P_{y'}Z_P^8 + 3R_yZ_R^6 + 9S_yZ_S^6 - 2\,\mathrm{d}SZ_S^5$$
$$- 2R_{y'}Z_R^8 - 4S_{y'}Z_S^8 - 4PZ_P^7 + Q\big(12Z_P^9 - Z_Q^7\big) + R\big(2Z_R^7 + 8Z_Q^9\big) + S\big(5Z_S^7 + 4Z_R^9\big)$$
$$+ 4\partial_{Q_{xy'y'}} + 18\partial_{P_{y'y'y'}},$$

$$\bar{X}_{29} = y'^3X_{26} - y'^2X_{27} + y'X_{28} - X_{29}(w_{yyyy'})$$
$$= -Z_R^2 + \mathrm{d}PZ_P^6 + \mathrm{d}QZ_Q^6 + \mathrm{d}RZ_R^6 + \mathrm{d}SZ_S^6 - 4PZ_P^8 - QZ_Q^8 + 2RZ_R^8 + 5SZ_S^8 + 4Z_Q^9,$$

$$X_{30}(w_{xxy'y'}) = \partial_{R_x} + \partial_{Q_{y'}} + P\big(\partial_{Q_{xx}} + 3\partial_{P_{xy'}}\big) + Q\big(2\partial_{R_{xx}} + \partial_{Q_{xy'}} - 3\partial_{P_{y'y'}}\big)$$
$$+ R\big(3\partial_{S_{xx}} - \partial_{R_{xy'}} - 2\partial_{Q_{y'y'}}\big) - S\big(3\partial_{S_{xy'}} + \partial_{R_{y'y'}}\big) + P_y\big(\partial_{Q_{xxy}} + 3\partial_{P_{xyy'}}\big)$$
$$+ 3P_x\partial_{Q_{xxx}} + (7P_x + y'P_y)\partial_{P_{xxy'}} + 6Q_x\partial_{R_{xxx}} + Q_y\big(2\partial_{R_{xxy}} + \partial_{Q_{xyy'}} - 3\partial_{P_{yy'y'}}\big)$$
$$+ (3Q_x + y'Q_y + P_{y'})\partial_{Q_{xxy'}} + (5P_{y'} - 3Q_x)\partial_{P_{xy'y'}} + 9R_x\partial_{S_{xxx}}$$
$$+ R_y\big(3\partial_{S_{xxy}} - \partial_{R_{xyy'}} - 2\partial_{Q_{yy'y'}}\big) + \big(Q_{y'} - 2R_x\big)\partial_{Q_{xy'y'}} + \big(2Q_{y'} - R_x + y'R_y\big)\partial_{R_{xxy'}}$$
$$- 9Q_{y'}\partial_{P_{y'y'y'}} + (3R_{y'} - 5S_x + y'S_y)\partial_{S_{xxy'}} - S_y\big(3\partial_{S_{xyy'}} + \partial_{R_{yy'y'}}\big)$$
$$- (S_x + 3R_{y'})\partial_{R_{xy'y'}} - 6R_{y'}\partial_{Q_{y'y'y'}} - S_{y'}\big(7\partial_{S_{xy'y'}} + 3\partial_{R_{y'y'y'}}\big),$$

$$\bar{X}_{31} = 2y'X_{30} - X_{31}(w_{xyy'y'}) = Z_R^0 + P\big(Z_Q^1 + 3Z_P^3\big) + Q\big(2Z_R^1 + Z_Q^3\big) + R\big(3Z_S^1 - Z_R^3\big)$$
$$- 3SZ_S^3 - 2\partial_{R_{xy'}} - 5\partial_{Q_{y'y'}} - P_y\big(2Z_Q^5 + 6Z_P^8\big) + \mathrm{d}P\big(2Z_Q^4 + 4Z_P^7\big) + 4\,\mathrm{d}QZ_R^4$$
$$- Q_y\big(4Z_R^5 + 2Z_Q^8\big) + 5P_{y'}Z_P^9 + (2\,\mathrm{d}Q + P_{y'})Z_Q^7 + R_y\big(2Z_R^8 - 6Z_S^5\big) + 6\,\mathrm{d}RZ_S^4$$
$$+ Q_{y'}\big(2Z_R^7 + Z_Q^9\big) + 6S_yZ_S^8 + (3R_{y'} - 2\,\mathrm{d}S)Z_S^7 - 3R_{y'}Z_R^9 - 7S_{y'}Z_S^9 - 7P\partial_{P_{xy'y'}}$$
$$+ Q\big(18\partial_{P_{y'y'y'}} - 2\partial_{Q_{xy'y'}}\big) + R\big(3\partial_{R_{xy'y'}} + 12\partial_{Q_{y'y'y'}}\big) + S\big(8\partial_{S_{xy'y'}} + 6\partial_{R_{y'y'y'}}\big),$$

$$\bar{X}_{32} = y'^2X_{30} - y'X_{31} + X_{32}(w_{yyy'y'}) = PZ_Q^2 + 2QZ_R^2 + 3RZ_S^2 - 2Z_R^3 - 3P_yZ_Q^6$$
$$+ \mathrm{d}P\big(Z_Q^5 + Z_P^8\big) - 6Q_yZ_R^6 + 2\,\mathrm{d}QZ_R^5 + (\mathrm{d}Q + P_{y'})Z_Q^8 - 9R_yZ_S^6 + (\mathrm{d}R + 2Q_{y'})Z_R^8$$
$$+ 3\,\mathrm{d}RZ_S^5 + (\mathrm{d}S + 3R_{y'})Z_S^8 - 7PZ_P^9 - 2QZ_Q^9 + 3RZ_R^9 + 8SZ_S^9 + 9\partial_{Q_{y'y'y'}},$$

$$X_{33}(w_{xy'y'y'}) = \partial_{S_x} + \partial_{R_{y'}} + P\big(\partial_{Q_{xy'}} + 3\partial_{P_{y'y'}}\big) + Q\big(2\partial_{R_{xy'}} + \partial_{Q_{y'y'}}\big) - 3S\partial_{S_{y'y'}}$$
$$+ R\big(3\partial_{S_{xy'}} - \partial_{R_{y'y'}}\big) + 2P_x\partial_{Q_{xxy'}} + P_y(\partial_{Q_{xyy'}} + 3\partial_{P_{yy'y'}}) + (4P_x + y'P_y)\partial_{P_{xy'y'}}$$
$$+ 4Q_x\partial_{R_{xxy'}} + Q_y\big(2\partial_{R_{xyy'}} + \partial_{Q_{yy'y'}}\big) + (2Q_x + y'Q_y + 2P_{y'})\partial_{Q_{xy'y'}} + 8P_{y'}\partial_{P_{y'y'y'}}$$
$$+ 6R_x\partial_{S_{xxy'}} + R_y\big(3\partial_{S_{xyy'}} - \partial_{R_{yy'y'}}\big) + 2Q_{y'}\partial_{Q_{y'y'y'}} + \big(4Q_{y'} + y'R_y\big)\partial_{R_{xy'y'}}$$
$$+ (6R_{y'} - 2S_x + y'S_y)\partial_{S_{xy'y'}} - 3S_y\partial_{S_{yy'y'}} - 4R_{y'}\partial_{R_{y'y'y'}} - 10S_{y'}\partial_{S_{y'y'y'}},$$

$$\bar{X}_{34} = y'X_{33} - X_{34}(w_{yy'y'y'}) = Z_S^0 + PZ_Q^3 + 2QZ_R^3 + 3RZ_S^3 - 3\partial_{R_{y'y'}} - 2P_yZ_Q^8$$
$$+ \mathrm{d}P\big(Z_Q^7 + Z_P^9\big) - 4Q_yZ_R^8 + 2\,\mathrm{d}QZ_R^7 + (\mathrm{d}Q + 2P_{y'})Z_Q^9 - 6R_yZ_S^8 + (\mathrm{d}R + 4Q_{y'})Z_R^9$$
$$+ 3\,\mathrm{d}RZ_S^7 + (\mathrm{d}S + 6R_{y'})Z_S^9 - 10P\partial_{P_{y'y'y'}} - 3Q\partial_{Q_{y'y'y'}} + 4R\partial_{R_{y'y'y'}} + 11S\partial_{S_{y'y'y'}},$$
$$X_{35}(w_{y'y'y'y'}) = \partial_{S_{y'}} + P\partial_{Q_{y'y'}} + 2Q\partial_{R_{y'y'}} + 3R\partial_{S_{y'y'}} + P_x\partial_{Q_{xy'y'}} + P_y\partial_{Q_{yy'y'}}$$
$$+ \mathrm{d}P\partial_{P_{y'y'y'}} + 2Q_x\partial_{R_{xy'y'}} + 2Q_y\partial_{R_{yy'y'}} + (\mathrm{d}Q + 3P_{y'})\partial_{Q_{y'y'y'}} + 3R_x\partial_{S_{xy'y'}}$$
$$+ 3R_y\partial_{S_{yy'y'}} + (\mathrm{d}R + 6Q_{y'})\partial_{R_{y'y'y'}} + (\mathrm{d}S + 9R_{y'})\partial_{S_{y'y'y'}},$$
$$X_{36}(w_{xxxxx}) = \partial_{P_{xx}} - 3Q\partial_{P_{xxx}} - 2R\partial_{Q_{xxx}} - S\partial_{R_{xxx}},$$
$$\bar{X}_{37} = 5y'X_{36} - X_{37}(w_{xxxxy}) = Z_P^1 - 3QZ_P^4 - 2RZ_Q^4 - SZ_R^4 - \partial_{Q_{xxx}} - 3\partial_{P_{xxy'}},$$
$$\bar{X}_{38} = 10y'^2X_{36} - 4y'X_{37} + X_{38}(w_{xxxyy}) = Z_P^2 - 3QZ_P^5 - 2RZ_Q^5 - SZ_R^5 - Z_Q^4 - 3Z_P^7,$$
$$\bar{X}_{39} = 10y'^3X_{36} - 6y'^2X_{37} + 3y'X_{38} - X_{39}(w_{xxyyy})$$
$$= -3QZ_P^6 - 2RZ_Q^6 - SZ_R^6 - Z_Q^5 - 3Z_P^8,$$
$$\bar{X}_{40} = 5y'^4X_{36} - 4y'^3X_{37} + 3y'^2X_{38} - 2y'X_{39} + X_{40}(w_{xyyyy}) = -Z_Q^6,$$
$$X_{41}(w_{yyyyy}) = y'X_{40} - y'^2X_{39} + y'^3X_{38} - y'^4X_{37} + y'^5X_{36},$$
$$X_{42}(w_{xxxxy'}) = \partial_{Q_{xx}} + \partial_{P_{xy'}} + 3P\partial_{P_{xxx}} + Q\big(\partial_{Q_{xxx}} - 3\partial_{P_{xxy'}}\big) - R\big(\partial_{R_{xxx}} + 2\partial_{Q_{xxy'}}\big)$$
$$- S(3\partial_{S_{xxx}} + \partial_{R_{xxy'}}),$$
$$\bar{X}_{43} = 4y'X_{42} - X_{43}(w_{xxxyy'}) = Z_Q^1 + Z_P^3 + 3PZ_P^4$$
$$+ Q\big(Z_Q^4 - 3Z_P^7\big) - R\big(Z_R^4 + 2Z_Q^7\big) - S\big(3Z_S^4 + Z_R^7\big) - \partial_{R_{xxx}} - 3\partial_{Q_{xxy'}} - 6\partial_{P_{xy'y'}},$$
$$\bar{X}_{44} = 6y'^2X_{42} - 3y'X_{43} + X_{44}(w_{xxyyy'}) = Z_Q^2 + 3PZ_P^5 + Q\big(Z_Q^5 - 3Z_P^8\big)$$
$$- R\big(Z_R^5 + 2Z_Q^8\big) - S\big(3Z_S^5 + Z_R^8\big) - Z_R^4 - 3Z_Q^7 - 6Z_P^9,$$
$$\bar{X}_{45} = 4y'^3X_{42} - 3y'^2X_{43} + 2y'X_{44} - X_{45}(w_{xyyyy'}) = 3PZ_P^6 + QZ_Q^6$$
$$- RZ_R^6 - 3SZ_S^6 - Z_R^5 - 3Z_Q^8,$$
$$\bar{X}_{46} = y'^4X_{42} - y'^3X_{43} + y'^2X_{44} - y'X_{45} + X_{46}(w_{yyyyy'}) = -Z_R^6,$$
$$X_{47}(w_{xxxy'y'}) = \partial_{R_{xx}} + \partial_{Q_{xy'}} + \partial_{P_{y'y'}} + P\big(\partial_{Q_{xxx}} + 3\partial_{P_{xxy'}}\big) - S\big(3\partial_{S_{xxy'}} + \partial_{R_{xy'y'}}\big)$$
$$+ Q\big(2\partial_{R_{xxx}} + \partial_{Q_{xxy'}} - 3\partial_{P_{xy'y'}}\big) + R\big(3\partial_{S_{xxx}} - \partial_{R_{xxy'}} - 2\partial_{Q_{xy'y'}}\big),$$
$$\bar{X}_{48} = 3y'X_{47} - X_{48}(w_{xxyy'y'}) = Z_R^1 + Z_Q^3 + P\big(Z_Q^4 + 3Z_P^7\big) + Q\big(2Z_R^4 + Z_Q^7 - 3Z_P^9\big)$$
$$+ R\big(3Z_S^4 - Z_R^7 - 2Z_Q^9\big) - S\big(3Z_S^7 + Z_R^9\big) - 2\partial_{R_{xxy'}} - 5\partial_{Q_{xy'y'}} - 9\partial_{P_{y'y'y'}},$$
$$\bar{X}_{49} = 3y'^2X_{47} - 2y'X_{48} + X_{49}(w_{xyyy'y'}) = Z_R^2 + P\big(Z_Q^5 + 3Z_P^8\big) + Q\big(2Z_R^5 + Z_Q^8\big)$$
$$+ R\big(3Z_S^5 - Z_R^8\big) - 3SZ_S^8 - 2Z_R^7 - 5Z_Q^9,$$
$$\bar{X}_{50} = y'^3X_{47} - y'^2X_{48} + y'X_{49} - X_{50}(w_{yyyy'y'}) = PZ_Q^6 + 2QZ_R^6 + 3RZ_S^6 - 2Z_R^8,$$
$$X_{51}(w_{xxy'y'y'}) = \partial_{S_{xx}} + \partial_{R_{xy'}} + \partial_{Q_{y'y'}} + P\big(\partial_{Q_{xxy'}} + 3\partial_{P_{xy'y'}}\big) - S\big(3\partial_{S_{xy'y'}} + \partial_{R_{y'y'y'}}\big)$$
$$+ Q\big(2\partial_{R_{xxy'}} + \partial_{Q_{xy'y'}} - 3\partial_{P_{y'y'y'}}\big) + R\big(3\partial_{S_{xxy'}} - \partial_{R_{xy'y'}} - 2\partial_{Q_{y'y'y'}}\big),$$
$$\bar{X}_{52} = 2y'X_{51} - X_{52}(w_{xyy'y'y'}) = Z_S^1 + Z_R^3 + P\big(Z_Q^7 + 3Z_P^9\big) + Q\big(2Z_R^7 + Z_Q^9\big)$$
$$+ R\big(3Z_S^7 - Z_R^9\big) - 3SZ_S^9 - 3\partial_{R_{xy'y'}} - 7\partial_{Q_{y'y'y'}},$$
$$\bar{X}_{53} = y'^2X_{51} - y'X_{52} + X_{53}(w_{yyy'y'y'}) = Z_S^2 + PZ_Q^8 + 2QZ_R^8 + 3RZ_S^8 - 3Z_R^9,$$
$$X_{54}(w_{xy'y'y'y'}) = \partial_{S_{xy'}} + \partial_{R_{y'y'}} + P\big(\partial_{Q_{xy'y'}} + 3\partial_{P_{y'y'y'}}\big) + Q\big(2\partial_{R_{xy'y'}} + \partial_{Q_{y'y'y'}}\big)$$
$$+ R\big(3\partial_{S_{xy'y'}} - \partial_{R_{y'y'y'}}\big) - 3S\partial_{S_{y'y'y'}},$$

$$\bar{X}_{55} = y'X_{54} - X_{55}(w_{yy'y'y'y'}) = Z_S^3 + PZ_Q^9 + 2QZ_R^9 + 3RZ_S^9 - 4\partial_{R_{y'y'y'}},$$
$$X_{56}(w_{y'y'y'y'y'}) = \partial_{S_{y'y'}} + P\partial_{Q_{y'y'y'}} + 2Q\partial_{R_{y'y'y'}} + 3R\partial_{S_{y'y'y'}},$$
$$X_{57}(w_{xxxxxx}) = \partial_{P_{xxx}},$$
$$\bar{X}_{58} = 6y'X_{57} - X_{58}(w_{xxxxxy}) = Z_P^4,$$
$$\bar{X}_{59} = 15y'^2X_{57} - 5y'X_{58} + X_{59}(w_{xxxxyy}) = Z_P^5,$$
$$\bar{X}_{60} = 20y'^3X_{57} - 10y'^2X_{58} + 4y'X_{59} - X_{60}(w_{xxxyyy}) = Z_P^6,$$

$$X_{61}(w_{xxyyyy}) = 3y'X_{60} - 6y'^2X_{59} + 10y'^3X_{58} - 15y'^4X_{57},$$
$$X_{62}(w_{xyyyyy}) = 2y'X_{61} - 3y'^2X_{60} + 4y'^3X_{59} - 5y'^4X_{58} + 6y'^5X_{57},$$
$$X_{63}(w_{yyyyyy}) = y'X_{62} - y'^2X_{61} + y'^3X_{60} - y'^4X_{59} + y'^5X_{58} - y'^6X_{57},$$
$$X_{64}(w_{xxxxxy'}) = \partial_{Q_{xxx}} + \partial_{P_{xxy'}},$$
$$\bar{X}_{65} = 5y'X_{64} - X_{65}(w_{xxxxyy'}) = Z_Q^4 + Z_P^7,$$
$$\bar{X}_{66} = 10y'^2X_{64} - 4y'X_{65} + X_{66}(w_{xxxyyy'}) = Z_Q^5 + Z_P^8,$$
$$\bar{X}_{67} = 10y'^3X_{64} - 6y'^2X_{65} + 3y'X_{66} - X_{67}(w_{xxyyyy'}) = Z_Q^6,$$
$$X_{68}(w_{xyyyyy'}) = 2y'X_{67} - 3y'^2X_{66} + 4y'^3X_{65} - 5y'^4X_{64},$$
$$X_{69}(w_{yyyyyy'}) = y'X_{68} - y'^2X_{67} + y'^3X_{66} - y'^4X_{65} + y'^5X_{64},$$
$$X_{70}(w_{xxxxy'y'}) = \partial_{R_{xxx}} + \partial_{Q_{xxy'}} + \partial_{P_{xy'y'}},$$
$$\bar{X}_{71} = 4y'X_{70} - X_{71}(w_{xxxyy'y'}) = Z_R^4 + Z_Q^7 + Z_P^9,$$
$$\bar{X}_{72} = 6y'^2X_{70} - 3y'X_{71} + X_{72}(w_{xxyyy'y'}) = Z_R^5 + Z_Q^8,$$
$$\bar{X}_{73} = 4y'^3X_{70} - 3y'^2X_{71} + 2y'X_{72} - X_{73}(w_{xyyyy'y'}) = Z_R^6,$$
$$X_{74}(w_{yyyyy'y'}) = y'X_{73} - y'^2X_{72} + y'^3X_{71} - Z^4X_{70},$$
$$X_{75}(w_{xxxy'y'y'}) = \partial_{S_{xxx}} + \partial_{R_{xxy'}} + \partial_{Q_{xy'y'}} + \partial_{P_{y'y'y'}},$$
$$\bar{X}_{76} = 3y'X_{75} - X_{76}(w_{xxyy'y'y'}) = Z_S^4 + Z_R^7 + Z_Q^9,$$
$$\bar{X}_{77} = 3y'^2X_{75} - 2y'X_{76} + X_{77}(w_{xyyy'y'y'}) = Z_S^5 + Z_R^8,$$
$$\bar{X}_{78} = y'^3X_{75} - y'^2X_{76} + y'X_{77} - X_{78}(w_{yyyy'y'y'}) = Z_S^6,$$
$$X_{79}(w_{xxy'y'y'y'}) = \partial_{S_{xxy'}} + \partial_{R_{xy'y'}} + \partial_{Q_{y'y'y'}},$$
$$\bar{X}_{80} = 2y'X_{79} - X_{80}(w_{xyy'y'y'y'}) = Z_S^7 + Z_R^9,$$
$$\bar{X}_{81} = y'^2X_{79} - y'X_{80} + X_{81}(w_{yyy'y'y'y'}) = Z_S^8,$$
$$X_{82}(w_{xy'y'y'y'y'}) = \partial_{S_{xy'y'}} + \partial_{R_{y'y'y'}},$$
$$\bar{X}_{83} = y'X_{82} - X_{83}(w_{yy'y'y'y'y'}) = Z_S^9,$$
$$X_{84}(w_{y'y'y'y'y'y'}) = \partial_{S_{y'y'y'}}.$$

Appendix B. Maple program for obtaining the operators $X_1, \ldots, X_{84}$

To obtain all the operators $X_1, \ldots, X_{84}$ of system (16) we have used the Maple code given below. Here we denote $z = y'$ and $d = \partial$. For brevity in arrays Dif and T we write in detail only the sequences of derivatives for the function P. They are followed by dots, which should be replaced by similar expressions for the functions Q, R and S. In arrays A, B dots should be replaces by missing derivatives of w in accordance with the numeration of operators X_i in appendix A.

```
[> restart;
[> alias(w=w1(x, y, z), P=P1(x, y, z), Q=Q1(x, y, z), R=R1(x, y, z), S=S1(x, y, z));
```

B and A are the arrays of all derivatives of the function $w(x, y, y')$ up to the sixth order and their notations, respectively

```
[> B := array(1..84, [w, diff(w, x), diff(w, y), diff(w, z), diff(w, x, x), . . . ,
     diff(w, z, z, z, z, z, z)]);
[> A := array(1..84, [W, Wx, Wy, Wz, Wxx, . . . , Wzzzzzz]);
```

A1 is the set of substitutions of the form B[i]=A[i]

```
[> A1 := {} : for i from 1 to 84 do A2 := A1 union B[i] = A[i]; A1 := A2; end do :
```

Dif is the array of partial differentiations $\partial_x, \partial_y, \partial_{y'}, \partial_P, \partial_Q, \partial_R, \partial_S, \partial_{P_x}, \ldots, \partial_{S_{y'y'y'}}$ in the prolonged operator $\widetilde{X}$

```
[> Dif := array(1..83, [dx, dy, dz, dP, dQ, dR, dS, dPx, dPy, dPz, . . . , dPxx, dPxy,
    dPyy, dPxz, dPyz, dPzz, . . . , dPxxx, dPxxy, dPxyy, dPyyy, dPxxz, dPxyz,
    dPyyz, dPxzz, dPyzz, dPzzz, . . .]);
[> Dif1 := convert(Dif, set) :
```

T is the array of variables T in prolongation formula (14)

```
[> T := array(1..43, [x, y, z, P, Q, R, S, diff(P, x), diff(P, y), diff(P, z), . . . ,
    diff(P, x, x), diff(P, x, y), diff(P, y, y), diff(P, x, z), diff(P, y, z),
    diff(P, z, z), . . .]);
```

C is the array of coordinates of the prolonged operator $\widetilde{X}$

```
[> C := array(1..83) :
```

First 7 entries in the array C are defined by (5), (6)

```
[> C[1] := −diff(w, z);
[> C[2] := w − z∗diff(w, z);
[> C[3] := diff(w, x) + z∗diff(w, y);
[> C[4] := diff(w, x, x, x) + 3∗z∗diff(w, x, x, y) + 3∗zˆ2∗diff(w, x, y, y)
    +zˆ3∗diff(w, y, y, y) − 3∗Q∗(diff(w, x, x) + 2∗z∗diff(w, x, y) + zˆ2∗diff(w, y, y))
    +P∗(3∗diff(w, x, z) + 3∗z∗diff(w, y, z) + diff(w, y));
[> C[5] := diff(w, x, x, z) + 2∗z∗diff(w, x, y, z) + zˆ2∗diff(w, y, y, z) + diff(w, x, y)
    +z∗diff(w, y, y) − 2∗R∗(diff(w, x, x) + 2∗z∗diff(w, x, y) + zˆ2∗diff(w, y, y))
    +Q∗(diff(w, x, z) + z∗diff(w, y, z)) + P∗diff(w, z, z);
[> C[6] := diff(w, x, z, z) + z∗diff(w, y, z, z) + diff(w, y, z) − S∗(diff(w, x, x)
    +2∗z∗diff(w, x, y) + zˆ2∗diff(w, y, y)) − R∗(diff(w, x, z) + z∗diff(w, y, z)
    +diff(w, y)) + 2∗Q∗diff(w, z, z);
[> C[7] := diff(w, z, z, z) − S∗(3∗diff(w, x, z) + 3∗z∗diff(w, y, z) + 2∗diff(w, y))
    +3∗R∗diff(w, z, z);
```

Coefficients $\xi_x, \xi_y, \xi_{y'}$ in prolongation formula (14)

```
[> X := array(1..3, [diff(C[1], x), diff(C[1], y), diff(C[1], z)]);
```

Coefficients $\eta_x, \eta_y, \eta_{y'}$ in prolongation formula (14)

```
[> Y := array(1..3, [diff(C[2], x), diff(C[2], y), diff(C[2], z)]);
```

Coefficients $\zeta_x, \zeta_y, \zeta_{y'}$ in prolongation formula (14)

```
[> Z := array(1..3, [diff(C[3], x), diff(C[3], y), diff(C[3], z)]);
```

Extension of operator X to the first-order derivatives of P, Q, R, S

```
[> for i from 4 to 7 do for j from 1 to 3 do
   C[3*i − 5 + j] := diff(C[i], T[j]) − diff(T[i], x)*X[j] − diff(T[i], y)*Y[j]
   −diff(T[i], z)*Z[j]; end do; end do;
```

Extension of operator X to the second-order derivatives of P, Q, R, S

```
[> for k from 1 to 4 do for j from 1 to 3 do for i from 3*k + 5 to
   3*k + 4 + j do C[3*k + 8 + 2^(j − 1) + i] := diff(C[i], T[j]) − diff(T[i], x)*X[j]
   −diff(T[i], y)*Y[j] − diff(T[i], z)*Z[j]; end do; end do; end do;
```

Extension of operator X to the third-order derivatives of P, Q, R, S

```
[> for k from 1 to 4 do for j from 1 to 3 do for i from 6*k + 14 to
   6*k + 12 + 2^(j − 1) + j do C[4*k + 20 + (j − 1)^2 + i] := diff(C[i], T[j])
   −diff(T[i], x)*X[j] − diff(T[i], y)*Y[j] − diff(T[i], z)*Z[j];
   end do; end do; end do;
```

In array C the derivatives of $w(x, y, y')$ are replaced by their notations from array A

```
[> for i from 1 to 83 do C1[i] := subs(A1, C[i]); end do :
```

Display all the operators $X_1, \ldots, X_{84}$

```
[> for i from 1 to 84 do l1 := 0; for j from 1 to 83 do l2 := l1;
   l3 := coeff(C1[j], A[i], 1)*Dif[j]; l1 := l2 + l3; end do;
   print("Operator X", i, "is equal to", collect(l1, Dif1)); end do;
```

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